1.1 Sets of Real Numbers and the Cartesian Coordinate Plane

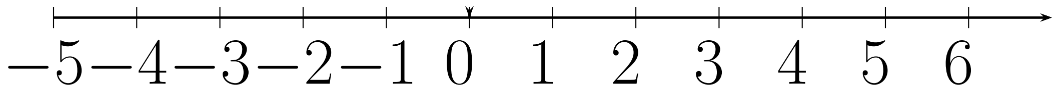
Sets and Interval Notation

Definition 1.1. Suppose A and B are two sets.

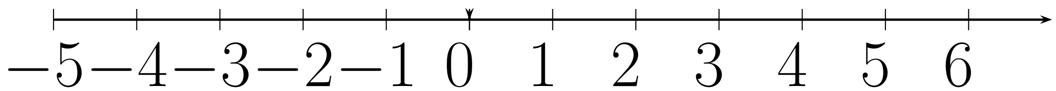
- The **intersection** of A and B: $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$
- The **union** of A and B: $A \cup B = \{x \,|\, x \in A \text{ or } x \in B \text{ (or both)}\}$

1.1.1 Interval and Inequality Notation, Numberlines

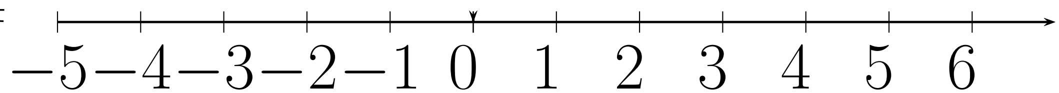
$x > 2$

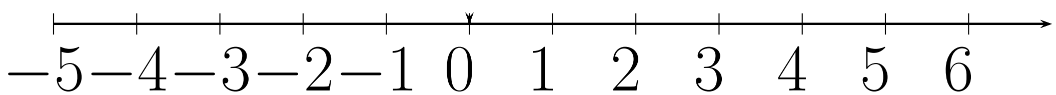


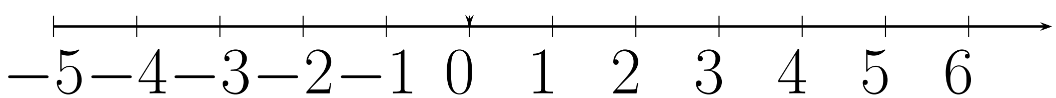
$x \geq 2$



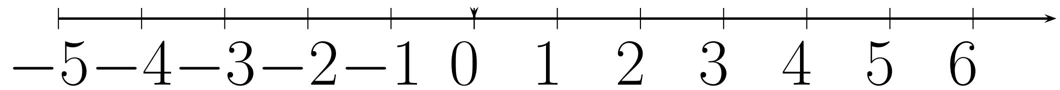
$x \geq 2$ AND $x \leq 4$

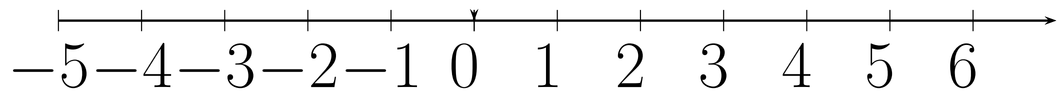


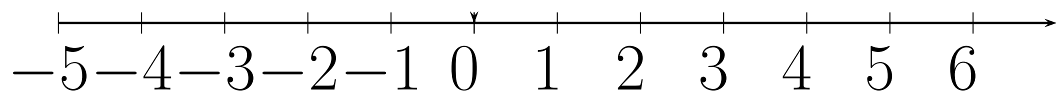




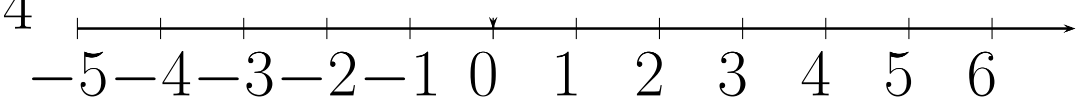
$x \leq 2$ OR $x \geq 4$







$x \leq 2$ AND $x \geq 4$

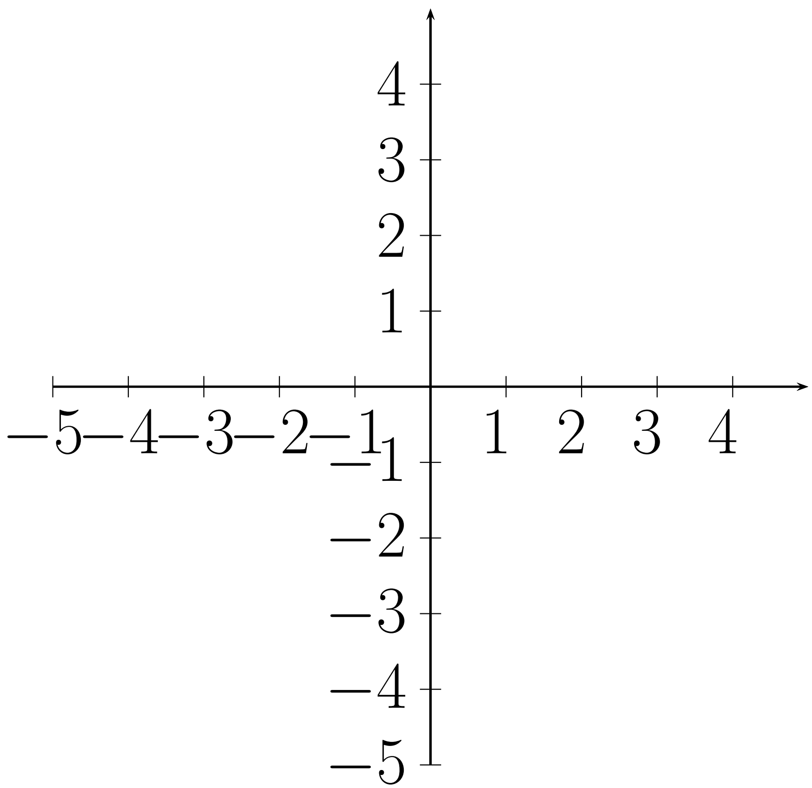


Interval Notation

Inequality notation		Interval notation
$x > 2$		$(2,\ \infty)$
$x \geq 2$		$[2,\ \infty)$
$x \geq 2$ AND $x \leq 4$	$2 \leq x \leq 4$	$[2,\ 4]$
$x \leq 2$ OR $x \geq 4$		$(-\infty, 2] \cup [4, \infty)$

1.1.2 Cartesian Coordinates and symmetry

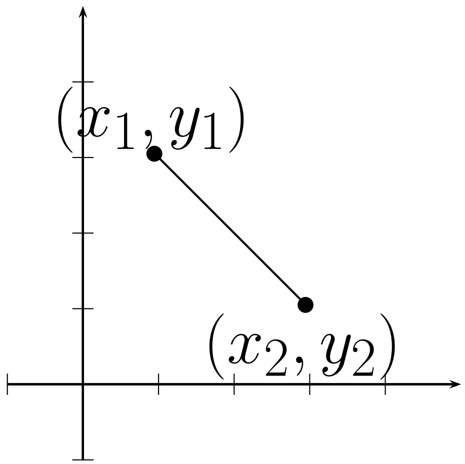
All points in the plane are ordered pairs (x, y) where the 1^{st} coordinate is directed distance on the x - axis and the 2^{nd} coordinate is directed distance on the y - axis. The xy-plane is divided into fours quadrants labeled I, II, III, and IV.



Example 1.1.1. At various times, the amount of water in a tub was measured and recorded in the table of values. Sketch a plot of the data.

Time (min)	Water in tub (gallons)
0	0
1	8
3	24
4	32

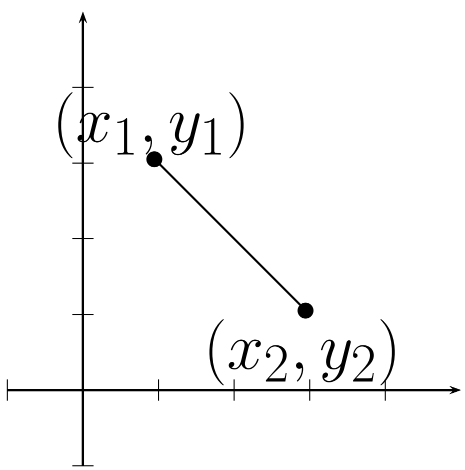
The distance Formula



The distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Midpoint Formula



The midpoint between two points is given by

$$\text{M.P.} = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$$

Example 1.1.2. Find the distance and midpoint between $P(3, -10)$ and $Q(-1, 2)$

Example 1.1.3. The midpoint of AB is at $(1, 5)$. If $A = (3, 7)$, find B.

Definition 1.2. Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the $\boldsymbol{x}$-axis** if $a = c$ and $b = -d$
- **symmetric about the $\boldsymbol{y}$-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Shifting Points (Reflections)

To reflect a point (x, y) about the:

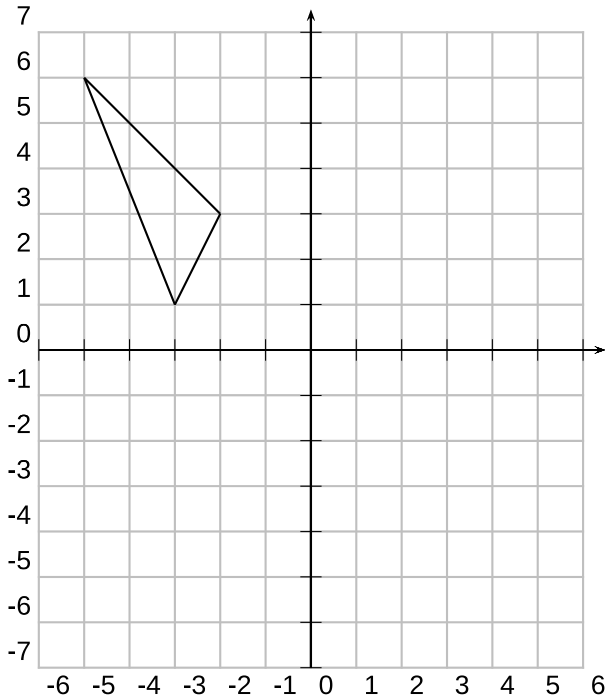
- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

Example 1.1.4. Use the graph below to

(1) Reflect the triangle over the x-axis.

(2) Reflect triangle over the y-axis.

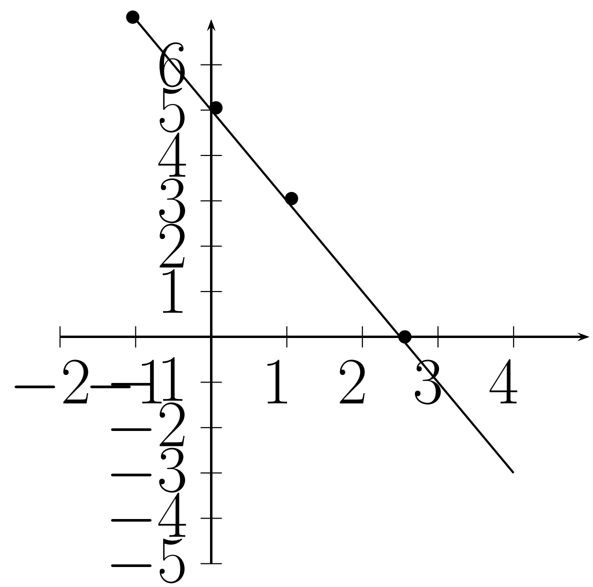
(3) Reflect triangle over the origin.



1.2 Relations

We reviewed in section 1.1 how to graph points so now we want to know how to graph equations. Suppose we want to graph the equation $y = -2x + 5$. This is a relationship between x and y where the value of y is determined by they choice of x. For each x we can find a y value and that is one point (x, y) on the graph:

x	$y = -2x + 5$
-1	(-2)(-1)+5 $= 7$
0	(-2)(0)+5 $= 5$
1	(-2)(1)+5 $= 3$
2	(-2)(2)+5 $= 1$
5/2	(-2)(5/2)+5 $= 0$



x and y Intercepts

x-intercept: The point where the graph crosses the x-axis.
To find the x-intercept you set $y = 0$.

y-intercept: The point where the graph crosses the y-axis.
To find the y-intercept you set $x = 0$.

Example 1.2.1. Find all intercepts for

$$y = 4x^3 - 16x$$

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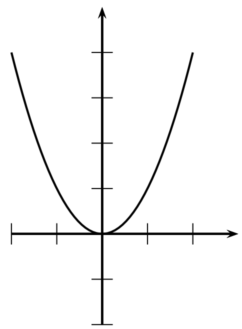
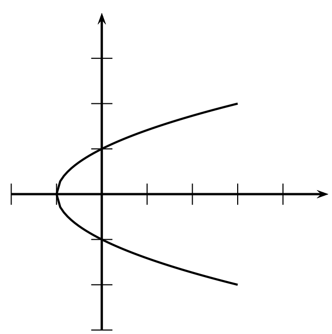
1.2.1 Symmetry



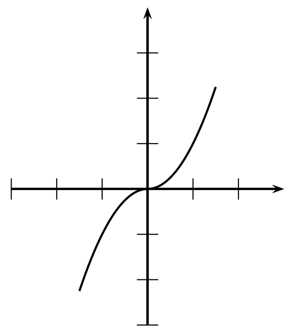
Symmetric about y-axis

A graph is symmetric about the y-axis if it is the same on both sides of the y-axis.
Thus when (a, b) is on the graph then $(-a, b)$ is also on the graph. $f(x) = f(-x)$ for all x.



Symmetric about x-axis

A graph is symmetric about the x-axis if it is the same on both sides of the x-axis.
Thus when (a, b) is on the graph then $(a, -b)$ is also on the graph.



Symmetric about the origin

A graph is symmetric about the origin if the graph is unchanged by a 180 degree rotation about the origin.

Thus when (a, b) is on the graph then $(-a, -b)$ is also on the graph.

The short version

Symmetry	The equation is equivalent when ...
y-axis	x is replaced with $-x$
x-axis	y is replace with $-y$
origin	x and y are replaced by $-x$ and $-y$.

Example 1.2.2. Find the symmetry of $y = x^3$.

Try replace x with $-x$

$$y = x^3 \qquad y = (-x)^3$$

Try replace y with $-y$

$$y = x^3 \qquad -y = (x)^3$$

Try replace x with $-x$ and y with $-y$

$$y = x^3 \qquad -y = (-x)^3$$

Draw a sketch: Since we have origin symmetry we can just plot a few positive numbers.

Example 1.2.3. Find the intercepts, determine if there is any symmetry and graph the function:

$$x^2 + y^3 = 1$$

Example 1.2.4. Find the intercepts, determine if there is any symmetry and graph the function:

$$x = 2y^3 + 3y$$

Example 1.2.5. Sketch the graph of $3x + 2y = -6$

1.3 Introduction to Functions

Definition 1.3. A **function** is a rule that establishes a correspondence between two sets of elements (called the **domain** and **range**) so that for every element in the domain there corresponds EXACTLY ONE element in the range.

Definition 1.4. A **function in one variable** is a set of ordered pairs with the property that no two ordered pairs have the same first element.

For example: { (-2, 1), (-1, 2), (0, 3), (1, 4), (2, 5) }.

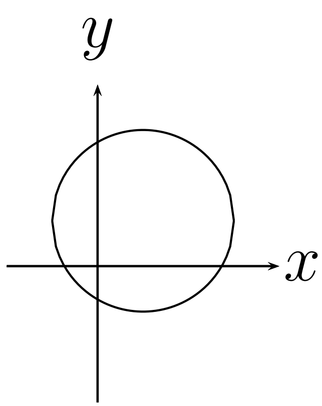
Definition 1.5. **Domain:** The "things" you can put into a function.

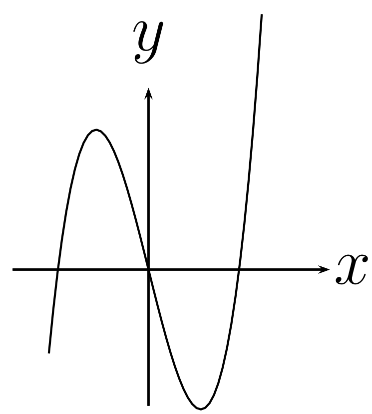
Range: The "things" you get out of a function.

Graphically

An equation defines a function if each vertical line drawn passes through the graph at most once. This is called the **Vertical Line Test**.

For example:





Example 1.3.1. Determine whether or not the relation represents y as a function of x. Find the domain and range of those relations which are functions.

1. $\{(-3, 9),\ (-2, 4),\ (-1, 1),\ (0, 0),\ (1, 1),\ (2, 4),\ (3, 9)\}$

2. $\{(-3, 0), (1, 6), (2, -3), (4, 2), (-5, 6), (4, -9), (6, 2)\}$

3. $\{(x, y)\,|\,x \text{ is an odd integer, and } y \text{ is an even integer}\}$

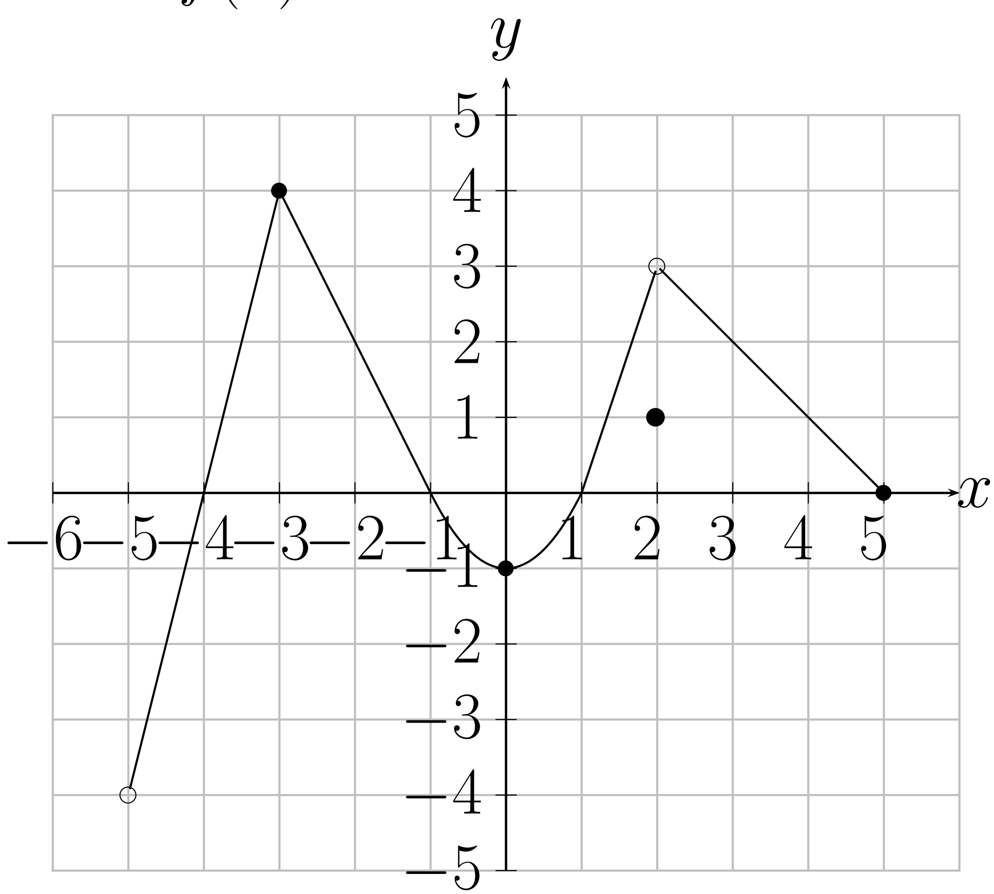
4. $\{(-2, y)\,|\,-3 < y < 4\}$

Example 1.3.2. Which of the following are functions of x and why?

1. $x^2 + y = 1$
2. $x + y^2 = 1$
3. $x + y^3 = 1$

Example 1.3.3. Find the domain and range of the function $y = \sqrt{x+8}$.

Example 1.3.4. Use the graph of $f(x)$ below to answer the questions about $f(x)$.



1. Find the domain of f.

2. Find the range of f.

3. Determine $f(2)$.

4. List the x-intercept(s), if any exist.

5. List the y-intercept(s), if any exist.

1.4 Function Notation

We can write a function several ways. The variable used to represent elements of the Domain is the **independent variable** and the variable used to represent elements of the Range is called the **dependent variable**. The most common way of writing a function is

$$\underbrace{y}_{\substack{\text{dependent}\\ \text{variable}}} = f(\underbrace{x}_{\substack{\text{independent}\\ \text{variable}}}) = \underbrace{2x+1}.$$

We can also write a function as

$$f : x \to 2x + 1 \qquad \text{OR} \qquad f : \{(x, y) \mid y = 2x + 1\}$$

Example 1.4.1. Consider the following function: $f(x) = 7x + 3$. Find the values of $f(0)$, $f(-1)$, $f(-1+h)$ and $f(x+h)$.

Example 1.4.2. Consider the following function:
$f(x) = 4x^2 + 3x - 22$. Find the values of $f(0)$, $f(-1)$, and $f(x+h)$.

Piecewise Functions

Example 1.4.3. Evaluate $f(0)$, $f(-1)$, $f(1)$, and $f(2)$ for

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x < 1 \\ 2x^2 + 2 & \text{if } x \geq 1 \end{cases}$$

Example 1.4.4. Determine the domain for the function. Write your answer in Interval Notation and as an Inequality.

$$f(x) = -1 + \sqrt{14x - 5}$$

Example 1.4.5. Determine the domain for the function. Write your answer in Interval Notation and as an Inequality.

$$f(x) = \frac{3x + 20}{x^2 + 8x - 8}$$

Example 1.4.6. Let $g(s) = \dfrac{s}{s+1} - 1$. Find the values of $g(10)$, $g\left(\frac{1}{11}\right)$, $g\left(\frac{-1}{8}\right)$

1.5 Function Arithmetic

Arithmetic Combinations

We can add, subtract, multiply and divide functions much like we do with real numbers.

Notation

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

Example 1.5.1. If $f(x) = 2x + 3$ and $g(x) = x^2 + 1$ find

$$(f + g)(x) \ = \ 2x + 3 + x^2 + 1 \ = \ x^2 + 2x + 4$$

$$(f - g)(x) =$$

$$(f \cdot g)(x) =$$

$$\left(\frac{f}{g}\right)(x) =$$

We can evaluate these new functions the exact same way we did before. Whatever is in the parentheses is replaced for x in the equation.

Example 1.5.2. If $f(x) = x^2 + 2x - 3$ and $g(x) = x^3 - 3x^2 - 4x$ find

a) $(f + g)(-1) =$

b) $(f \cdot g)(2) =$

c) The domain of $\left(\frac{f}{g}\right)(x) =$

Example 1.5.3. Suppose $f(x) = x^2 - 2x + 1$. Find

$$\frac{f(x+h) - f(x)}{h} \quad \text{(Difference Quotient)}$$

Example 1.5.4. A company produces very unusual CD's for which the variable cost is \$ 7 per CD and the fixed costs are \$ 30000. They will sell the CD's for \$ 52 each. Let x be the number of CD's produced.

1. Write the total cost C as a function of the number of CD's produced. $C(x)$
2. Write the total revenue R as a function of the number of CD's produced. $R(x)$
3. Write the total profit P as a function of the number of CD's produced. $P(x)$
4. Find the number of CD's which must be produced to break even.

1.6 Graphs of Functions

Definition 1.6. The **graph of a function** f is a collection of ordered pairs $(x, f(x))$ such that x is in the domain of $f(x)$.

Recall:

x is the distance in the x-direction. $y = f(x)$ is the the distance in the y direction.

Domain and Range

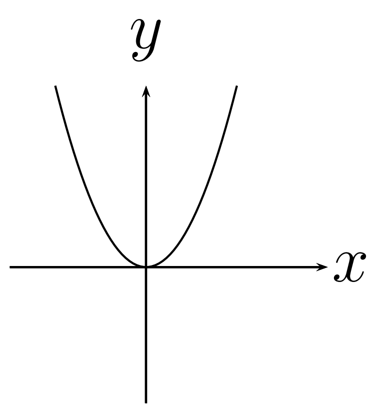
The **domain** of a function is those x-values that we can use in the function.

The **range** of a function is the y-values we get out of the function.

Example 1.6.1. $y = x^2$

Domain: All real numbers.

Range: $y \geq 0$.



Zeros of a Function

Definition 1.7. The **zeros of a function** $f(x)$ are those x-values for which $f(x) = 0$.

Q: How do we find the zeros of a function?

A: Set the function equal to zero. Also

Factor! Factor! Factor!

Example 1.6.2. Find the zeros of $f(x) = 3x^2 + 22x - 16$

Example 1.6.3. Find the zeros of $f(x) = \dfrac{x^2 - 9x + 14}{4x}$.

Increasing and Decreasing Functions

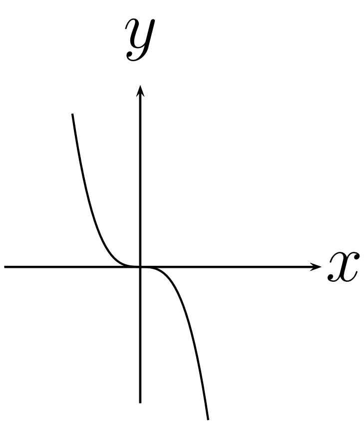
Definition 1.8.

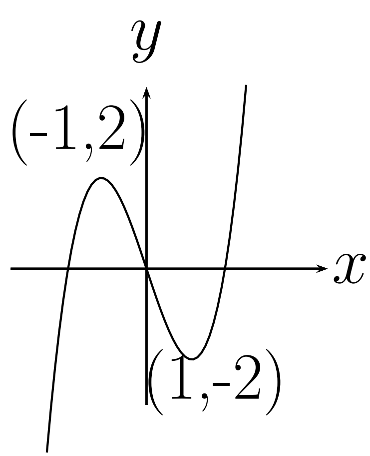
A function is **increasing** on an interval if for any x_1 and x_2 in the interval with $x_1 < x_2$ then $f(x_1) < f(x_2)$.

A function is **decreasing** on an interval if for any x_1 and x_2 in the interval with $x_1 < x_2$ then $f(x_1) > f(x_2)$.

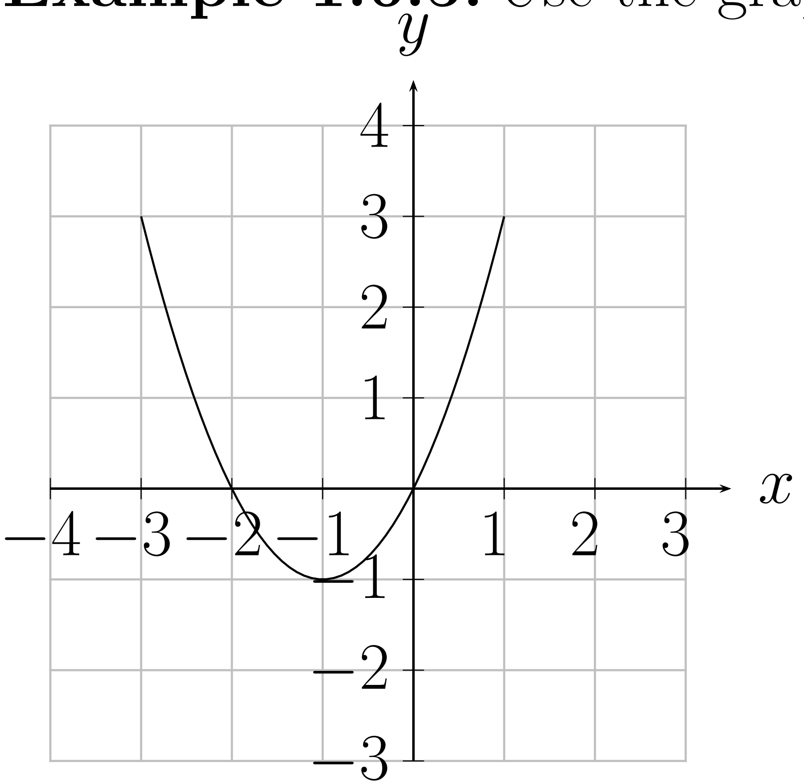
A function is **constant** on an interval if for any x_1 and x_2 in the interval $f(x_1) = f(x_2)$.

Example 1.6.4.





Example 1.6.5. Use the graph to solve the equation $x^2 + 2x = 0$



Linear Functions

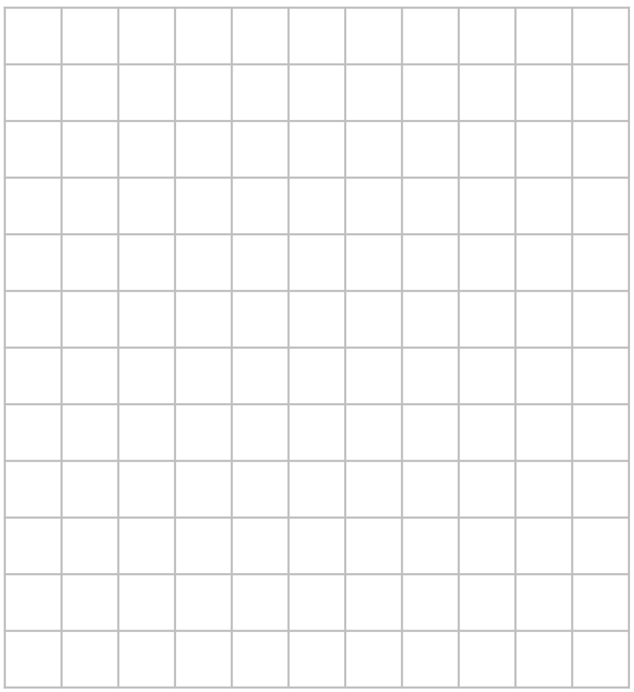
$$f(x) = mx + b \qquad \text{Linear Function}$$

Graph:

Example 1.6.6. Graph $f(x) = \frac{3}{2}x - 2$

Step 1: Plot y- intercept.

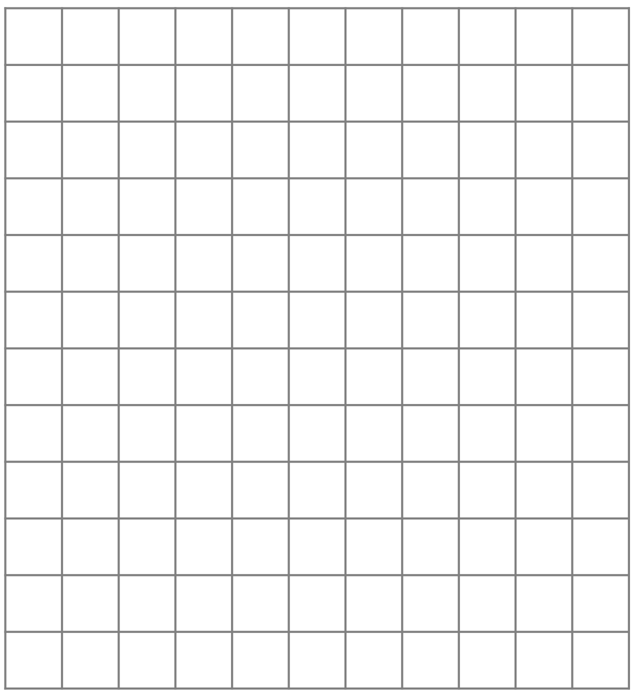
Step 2: Plot another point using the slope $\dfrac{3}{2} = \dfrac{\text{rise}}{\text{run}}$



Graphing Piecewise Functions

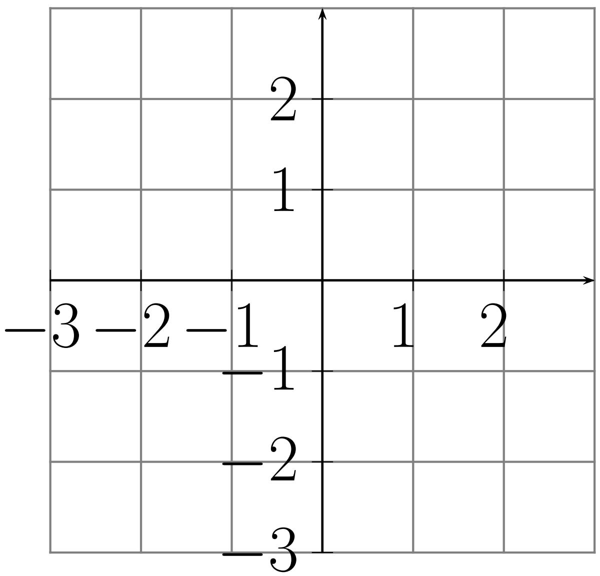
Example 1.6.7. Graph

$$f(x) = \begin{cases} \frac{3}{2}x - 2, & x \geq 2 \\ \frac{3}{2}x + 7, & x < 2 \end{cases}$$

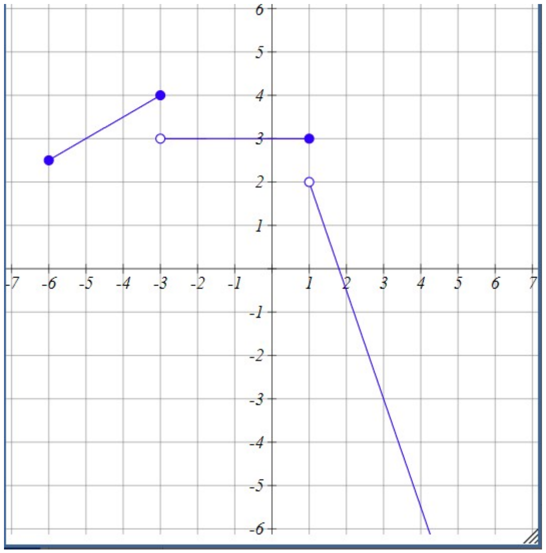


Example 1.6.8. Graph

$$f(x) = \begin{cases} -x, & x \leq 0 \\ 0, & 0 < x \leq 1 \\ x - 1, & x > 1 \end{cases}$$



Example 1.6.9. Write a piecewise function for the graph shown below.



Even and odd functions

Definition 1.9.

A function is **even** if $f(x) = f(-x)$ for all x in the domain of $f(x)$.

A function is **odd** if $-f(x) = f(-x)$ for all x in the domain of $f(x)$.

Example 1.6.10. Is $h(x) = x^5 - 5x^3$ even, odd or neither?

Look at $h(-x)$:

$$h(-x) = (-x)^5 - 5(-x)^3$$

Example 1.6.11. Is $h(x) = x^4 - 3x^2$ even, odd or neither?

Look at $h(-x)$:

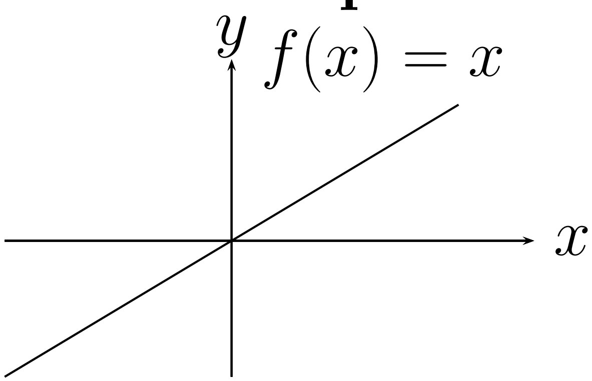
$$h(-x) = (-x)^4 - 3(-x)^2$$

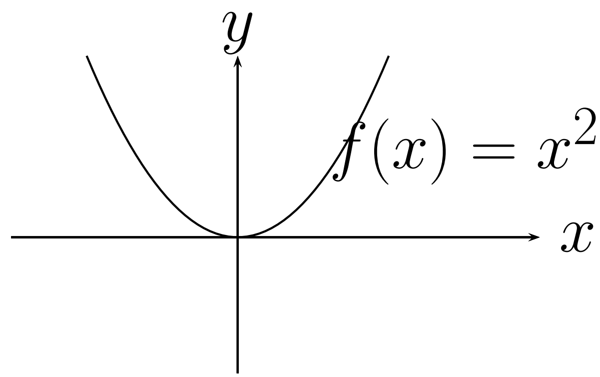
Example 1.6.12. Is $h(x) = x^3 - 5$ even, odd or neither?

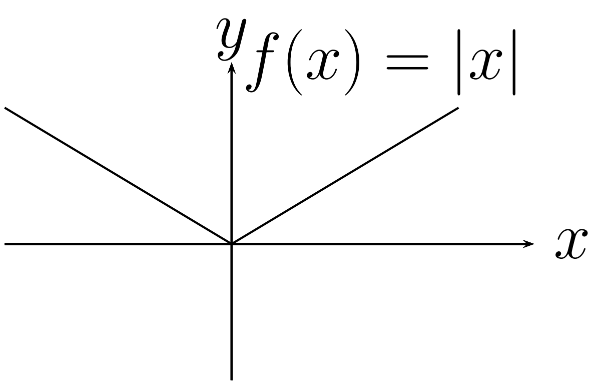
Look at $h(-x)$:

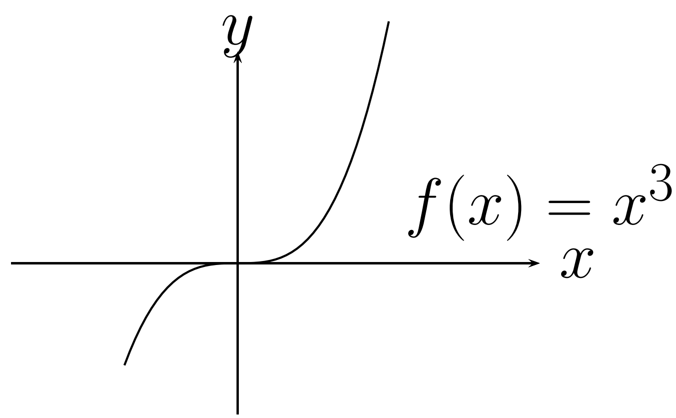
1.7 Transformations

Basic Graphs









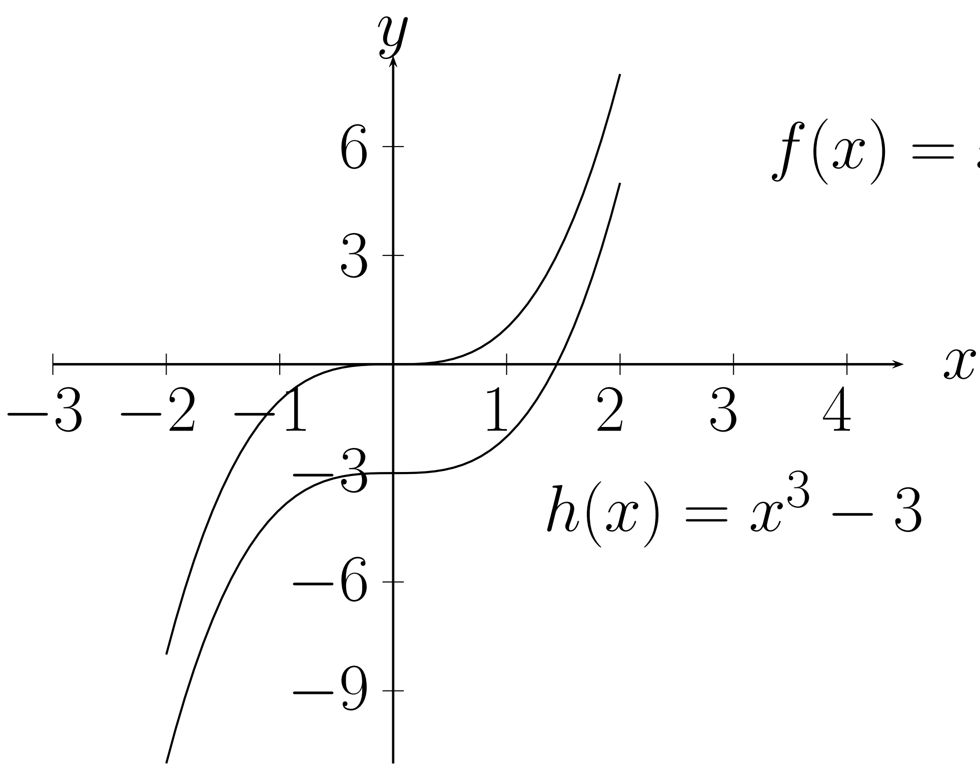
Shifting Graphs

Moving up and down

$h(x) = f(x) + a$ moves $f(x)$ up "a" units.

$h(x) = f(x) - a$ moves $f(x)$ down "a" units.

Example 1.7.1. $h(x) = x^3 - 3 = f(x) - 3$ if $f(x) = x^3$. Graph $f(x)$ and $h(x)$ on the same set of axes.

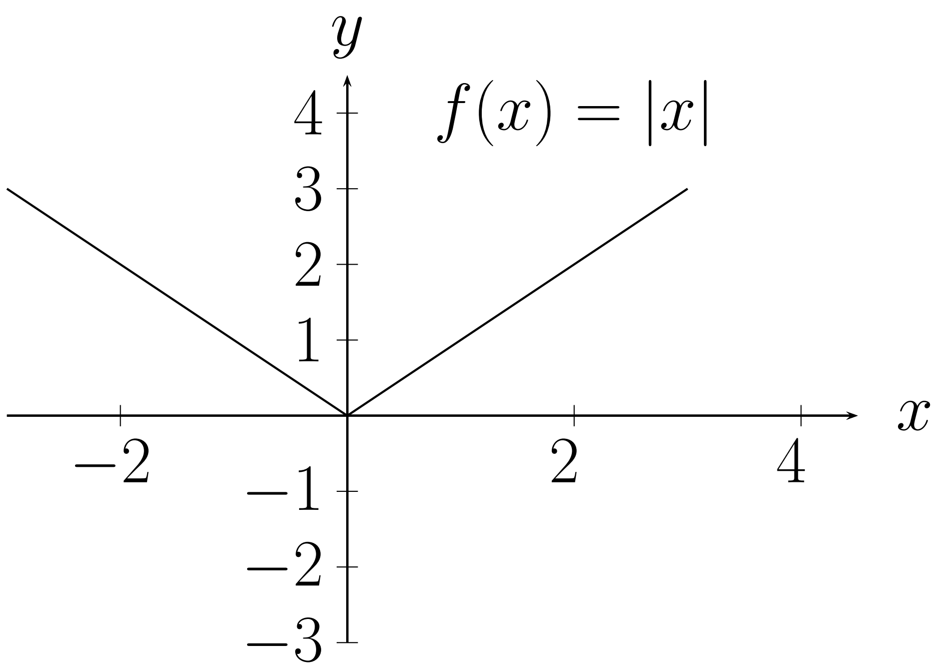


Moving left and right

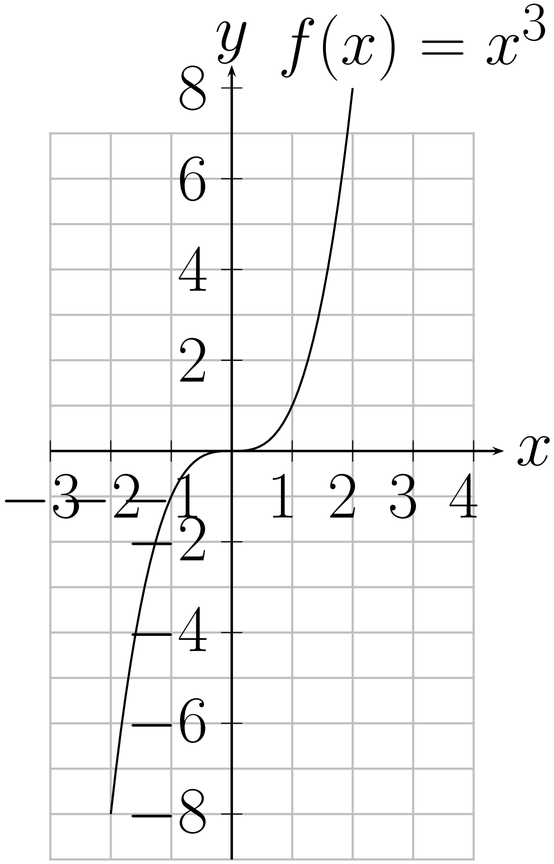
$h(x) = f(x + a)$ moves $f(x)$ to the left "a" units.

$h(x) = f(x - a)$ moves $f(x)$ to the right "a" units.

Example 1.7.2. $h(x) = |x - 2| = f(x - 2)$ if $f(x) = |x|$. Graph $f(x)$ and $h(x)$ on the same set of axes.



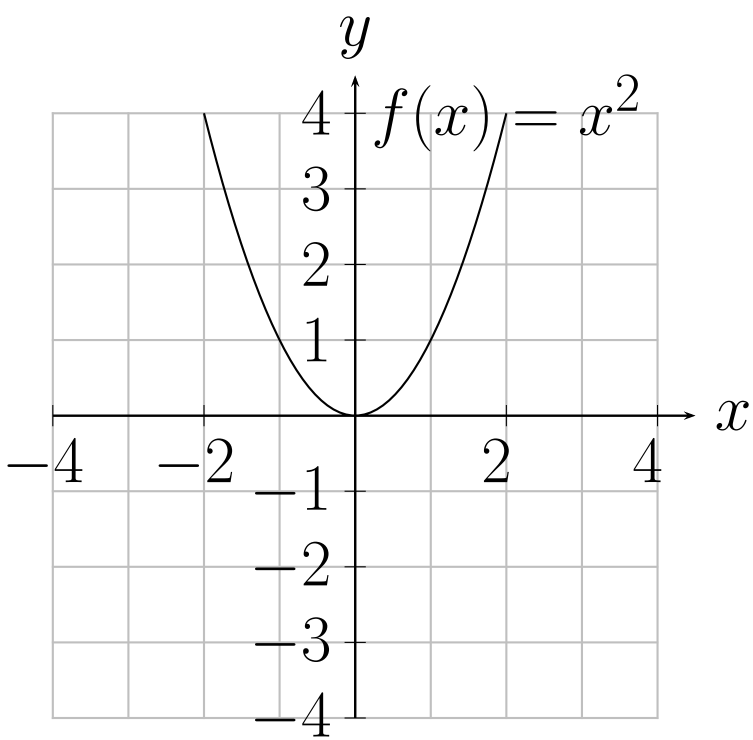
Example 1.7.3. $h(x) = (x+2)^3 - 3 = f(x+2) - 3$ if $f(x) = x^3$.
Graph $f(x)$ and $h(x)$ on the same set of axes.



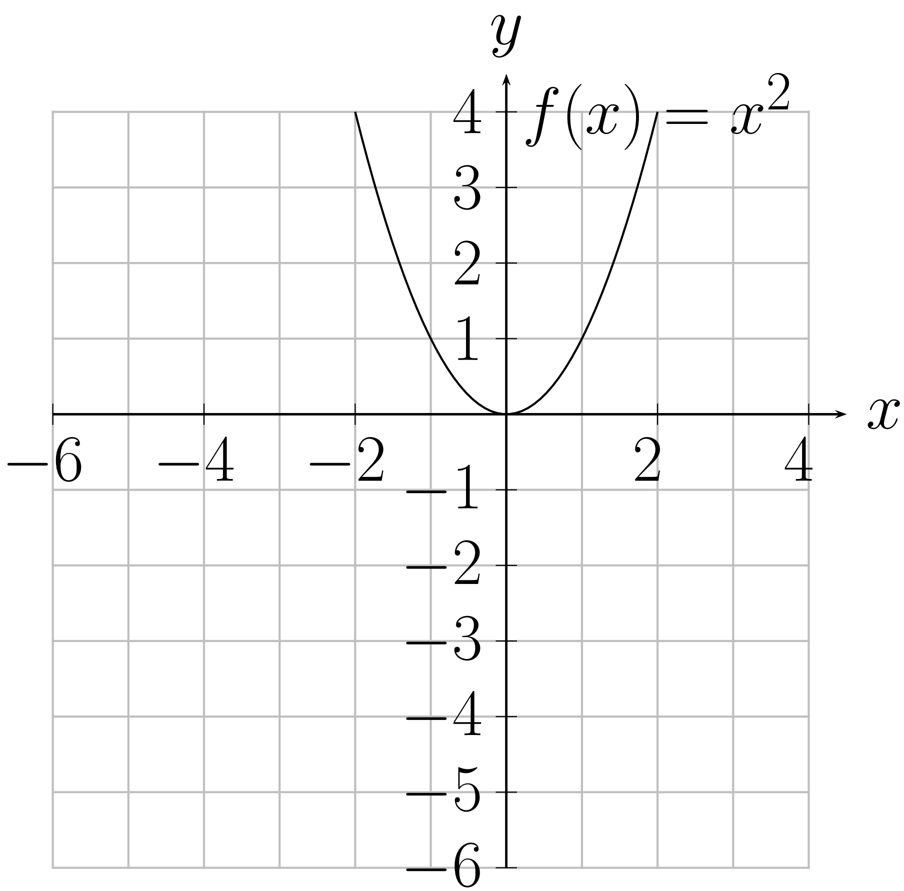
Reflecting across x-axis

$h(x) = -f(x)$ reflects $f(x)$ across the x-axis.

Example 1.7.4. $h(x) = -x^2 = -f(x)$ if $f(x) = x^2$. Graph $f(x)$ and $h(x)$ on the same set of axes.



Example 1.7.5. $h(x) = -(x+2)^2+2 = -f(x+2)+2$ if $f(x) = x^2$. Graph $f(x)$ and $h(x)$ on the same set of axes.



Vertical stretch or expansion

$$h(x) = A * f(x)$$

stretches $f(x)$ vertically if $A > 1$ and expands $f(x)$ horizontally if $0 < A < 1$.

Given the function

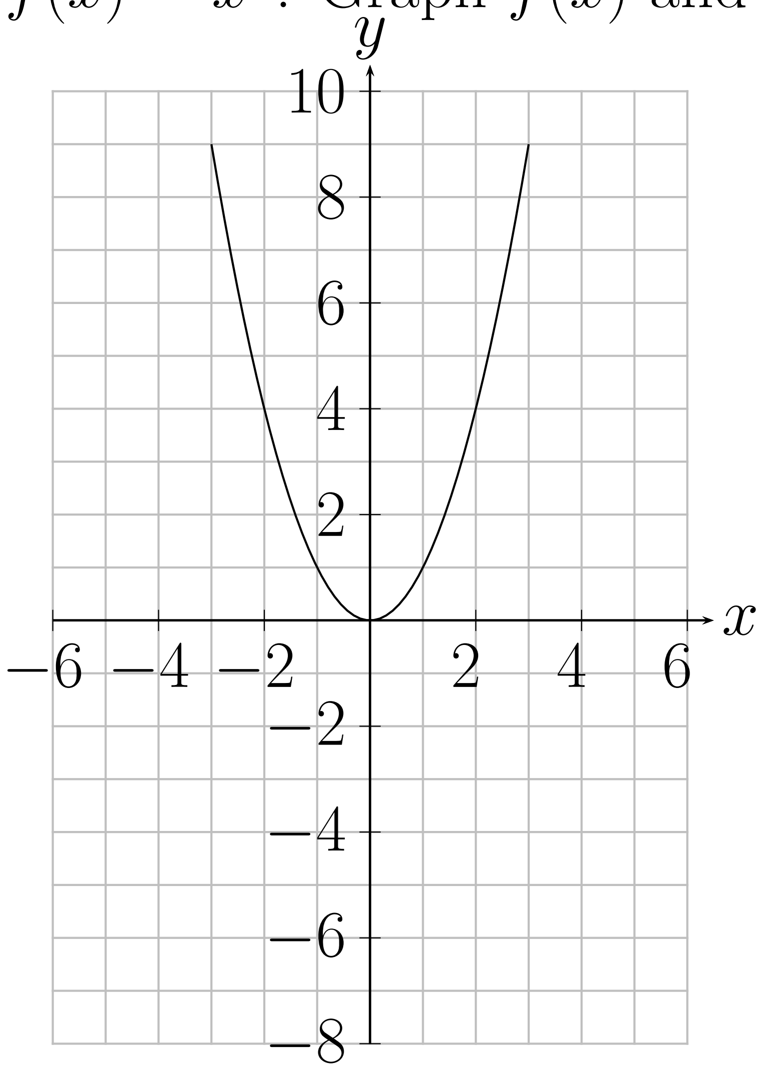
$$g(x) = Af\left(x + B\right) + C$$

the following transformations occur on $f(x)$:

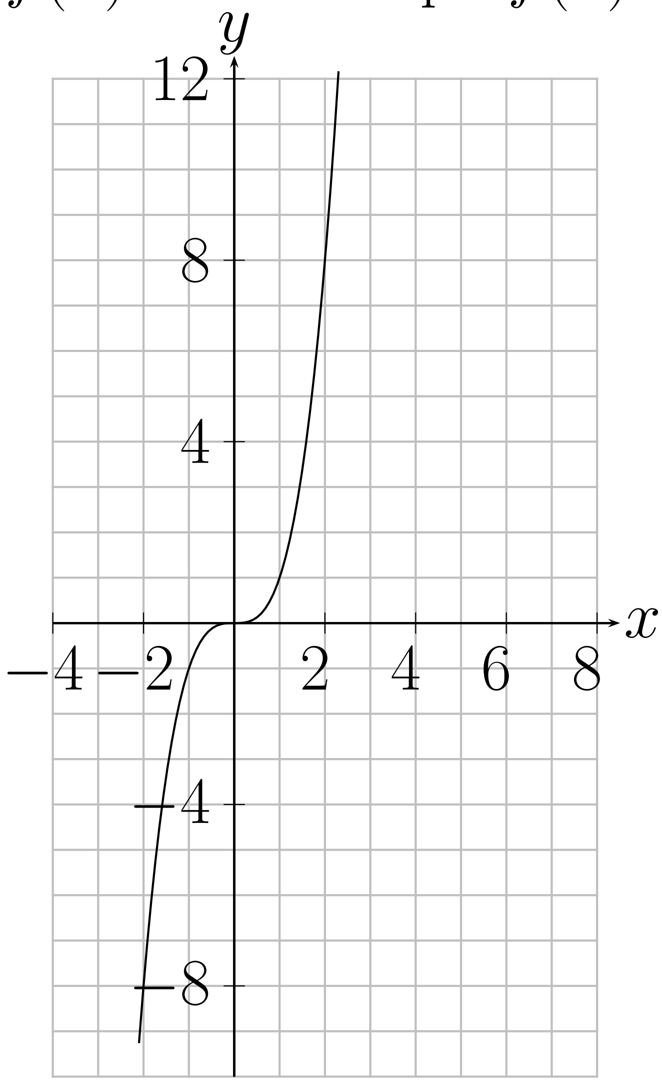
1. $|A|$ stretches or expands the function $f(x)$ by a factor $|A|$.
2. B moves the function $f(x)$ left $(B > 0)$ or right $(B < 0)$
3. C moves the function $f(x)$ up $(C > 0)$ or down $(C < 0)$

A negative sign in front of the function (A is negative) will reflect it over the x-axis.

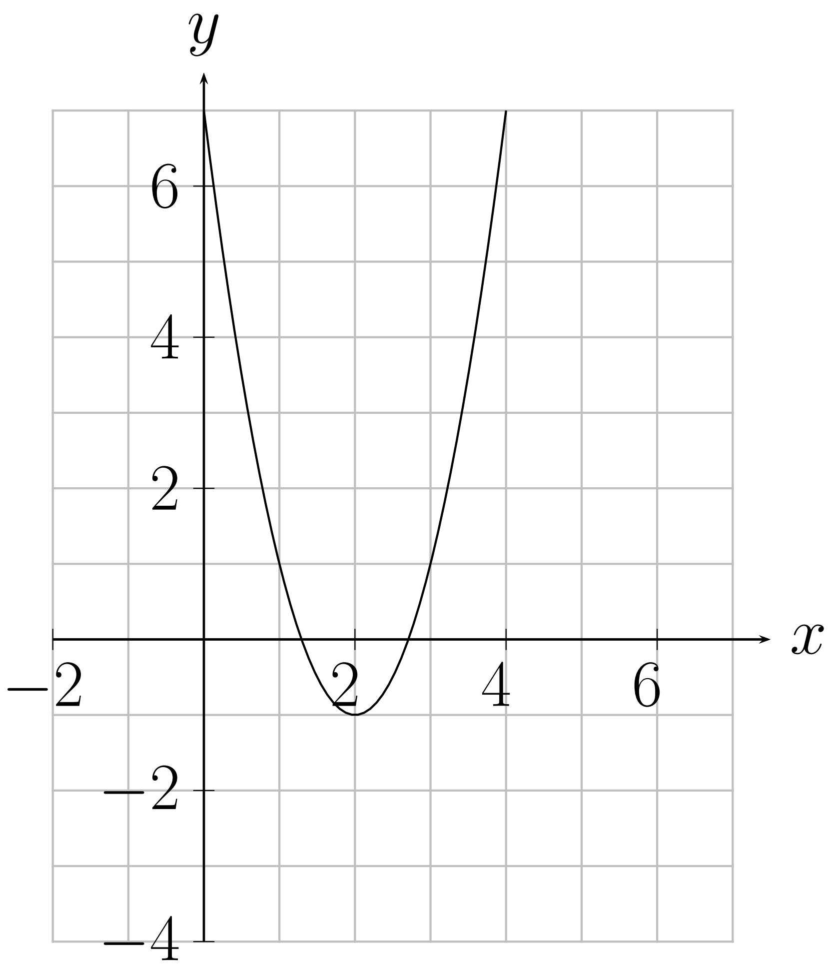
Example 1.7.6. $h(x) = 2(x+3)^2 - 3 = 2f(x+3) - 3$ where $f(x) = x^2$. Graph $f(x)$ and $h(x)$ on the same set of axes.



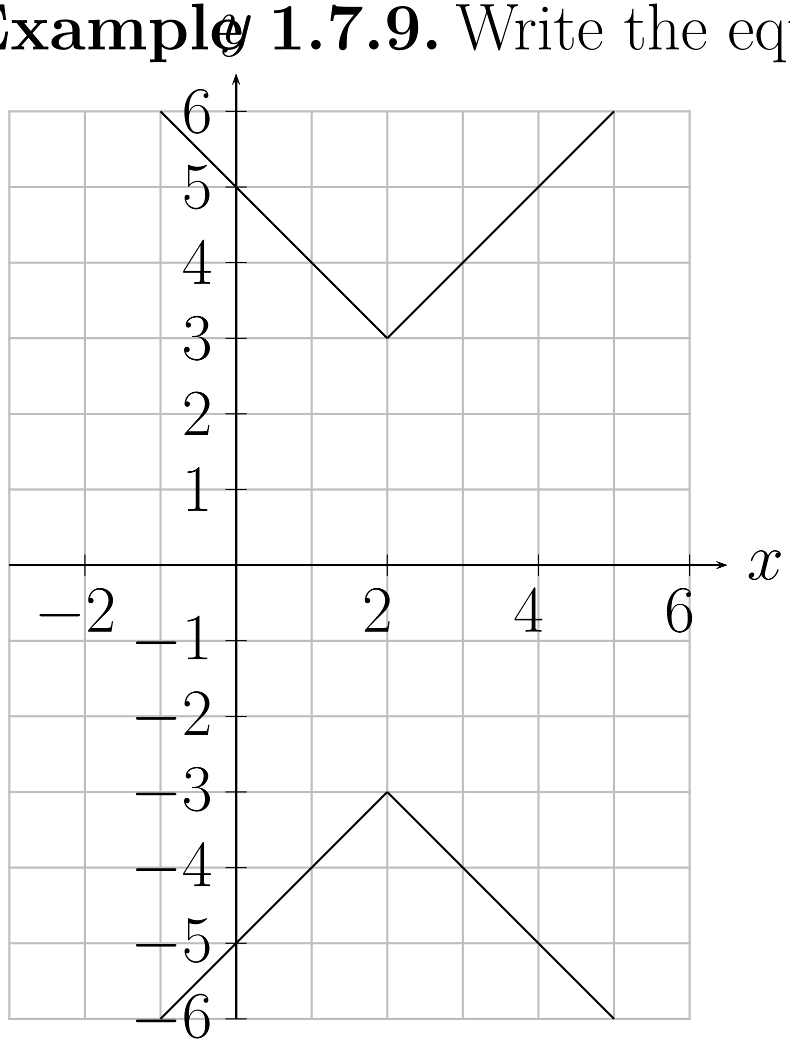
Example 1.7.7. $h(x) = -\frac{1}{2}(x-4)^3 + 2 = -\frac{1}{2}f(x-4) + 2$ if $f(x) = x^3$. Graph $f(x)$ and $h(x)$ on the same set of axes.



Example 1.7.8. Write the equations of the following graphs.



Example 1.7.9. Write the equations of the following graphs.



Example 1.7.10. Write the equation for the function that has the shape of $f(x) = x^2$ but is shifted 3 units to the left, 7 units up and then reflected across the x axis.