

## 6 Exponential and Logarithmic Functions

### 6.1 Introduction to Exponential and Logarithmic Functions

#### 6.1.1 Exponential Functions

##### The Exponential Function

The exponential function with base  $a$  is denoted by

$$f(x) = a^x$$

where  $a \neq 0$ ,  $a \neq 1$ , and  $x$  is any real number.

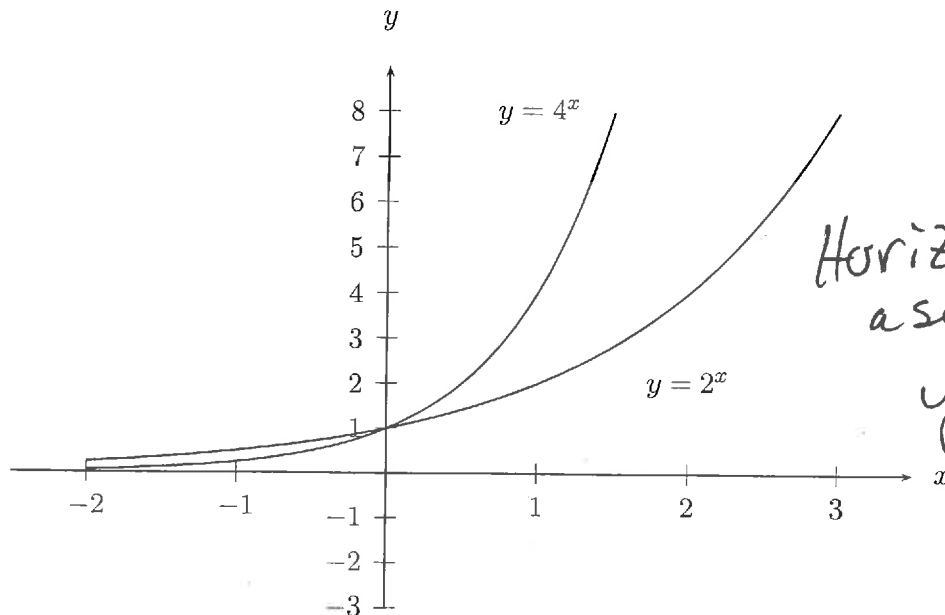
#### Graphs of exponential functions

Consider  $f(x) = 2^x$  and  $g(x) = 4^x$ .

Let's look at a table of  $x$  and  $y$  values for these functions.

$x$	$f(x)$	$g(x)$
-2	1/4	1/16
-1	1/2	1/4
0	1	1
1	2	4
2	4	16
3	8	64

We can graph this data on the  $xy$ -axes.



$$2^2 = 4$$

$$2^{1.75} \approx 3.36$$

$$\frac{2^{-2}}{1} = \frac{1}{2^2}$$

$$\pi^0 = 1$$

$$1.7^0 = 1$$

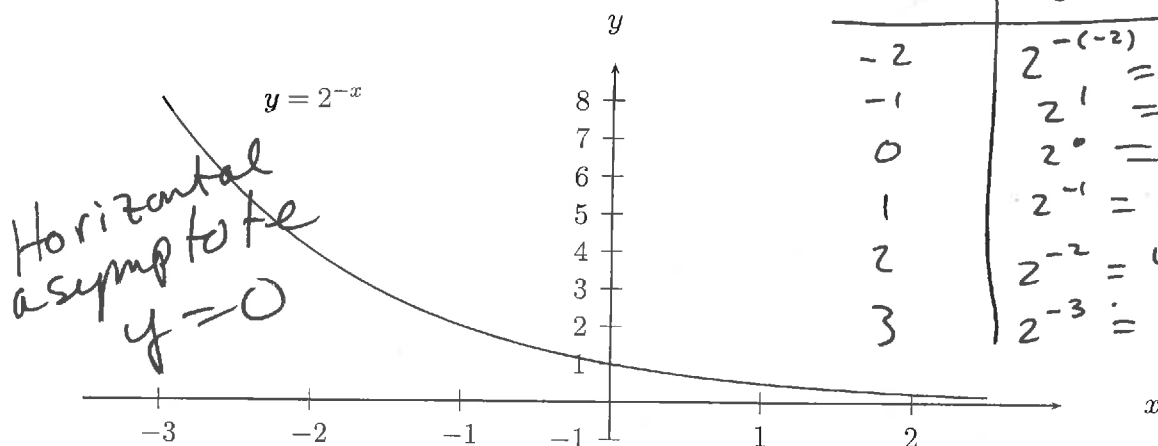
$$\frac{1}{2^{-2}} = \frac{2^2}{1}$$

$$(-1.7)^0 = 1$$

Anything to the zero power is 1

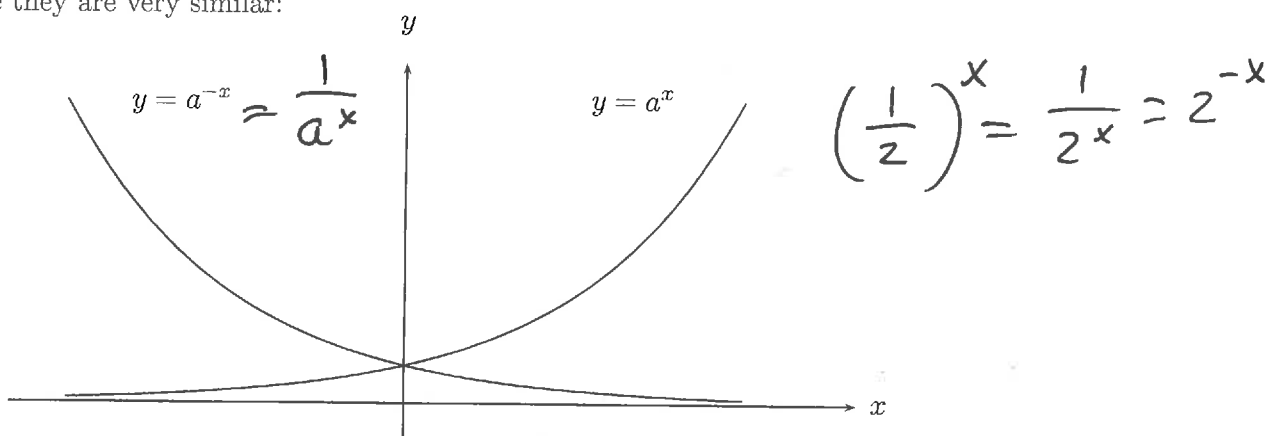
is 1

What about the graph of  $f(x) = 2^{-x}$ ?



$x$	$2^{-x}$
-2	$2^{-(-2)} = 2^2 = 4$
-1	$2^{-(-1)} = 2^1 = 2$
0	$2^{-0} = 1$
1	$2^{-1} = 1/2$
2	$2^{-2} = 1/4$
3	$2^{-3} = 1/8$

In general we have two basic shapes.  $y = a^x$  and  $y = a^{-x}$ . If we plot them on the same set of axes we can see they are very similar:



### The Natural Base $e$

#### The Number $e$

The following limits produce the same number and we call that number  $e$ .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$$

$$e \approx 2.71828 \dots$$

$e$  is just a number

Since  $e$  is a number we can use it in the exponential function  $f(x) = e^x$ . Why? Because it works in many situations.

**Finance:** Compound interest.

$P$  is some principal amount of money.

$r$  is the interest rate per year.

Simple interest:

After 1 year we have  $P_1 = P + rP = P(1+r)$

After 2 year we have  $P_2 = P_1 + rP_1 = P_1(1+r) = P(1+r)(1+r) = P(1+r)^2$

In fact at year  $t$  you have

$$P_t = P(1+r)^t$$

*Interest* 100 at 5% for 10 years

$$100(1+0.05)^{10} = 162$$

If you compound each month then you have to add  $1/12$  of the interest every month and you get

$$P_t = P \left(1 + \frac{r}{12}\right)^{12t}, \quad \text{where } t = \text{number of years. } 100 \left(1 + \frac{0.05}{12}\right)^{12 \cdot 10}$$

In general if you compound  $n$  times per year you have

$$P_t = P \left(1 + \frac{r}{n}\right)^{nt}$$

If we compound continuously that mean  $n \rightarrow \infty$  so

$$P_t = Pe^{rt}$$

$$= 164.70$$

**Example 6.1.1.** My credit card has \$4000 on it. The interest rate is 18% per year. If I make no payments how much do I owe at the end of three years if the credit card company compounds the interest (a) yearly, (b) monthly, (c) daily, and (d) continuously?

$$P = 4000$$

$$r = 0.18$$

$$(a) \quad t=3 \quad n=1$$

$$(b) \quad t=3 \quad n=12$$

$$(c) \quad t=3 \quad n=365$$

$$(d) \quad t=3$$

$$(a) \quad P_t = 4000(1+0.18)^3 = 6572.13$$

$$(b) \quad P_t = 4000 \left(1 + \frac{0.18}{12}\right)^{12 \cdot 3} = 6836.56$$

$$(c) \quad P_t = 4000 \left(1 + \frac{0.18}{365}\right)^{365 \cdot 3} = 6863.11$$

$$(d) \quad P_t = 4000 e^{(0.18)(3)} = 6864.03$$

## Properties of exponents

*Memorize*

Let  $a$  and  $b$  be positive numbers with  $a \neq 1$ ,  $b \neq 1$  and let  $x$  and  $y$  be real numbers. Then:

A) Exponent Laws:

$$1. a^x a^y = a^{x+y}$$

$$2. (a^x)^y = a^{xy}$$

$$3. (ab)^x = a^x b^x$$

$$4. \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$5. \frac{a^x}{a^y} = a^{x-y}$$

$$x^3 \cdot x^4 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$$

$$(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{12}$$

$$\begin{aligned} (ab)^3 &= a \cdot b \cdot a \cdot b \cdot a \cdot b \\ &= a \cdot a \cdot a \cdot b \cdot b \cdot b \\ &= a^3 \cdot b^3 \end{aligned}$$

B)  $a^x = a^y$  if and only if  $x = y$ .

$$\frac{a^x}{a^y} = a^x \cdot a^{-y} = a^{x-y}$$

~~Example 6.1.2. Solve  $64^{x+3} = 16$~~

$$1 = \frac{a^3}{a^3} = a^{3-3} = a^0$$

## 6.1.2 Logarithms

Let  $a > 0$ ,  $a \neq 1$  and  $x > 0$  then

$$y = \log_a(x) \iff x = a^y$$

The function  $f(x) = \log_a(x)$  is called the **logarithmic function with base  $a$**

OR

"log base  $a$ "

Example 6.1.3.

$$16^{3/4} = (16^{1/4})^3 = 2^3 = 8$$

1.  $\log_3(81) = 4 \iff 3^4 = 81$

2.  $\log_{16} 8 = \frac{3}{4} \iff 16^{3/4} = 8$

$$16^3 = (16)(16)(16)$$

3.  $10^{-3} = 0.001 \iff \log_{10}(0.001) = -3$

$$16^{1/4} = \sqrt[4]{16}$$

Example 6.1.4. Find  $y$ .  $\log_5(125) = y$

$$5^y = 125$$

$$\boxed{y = 3}$$

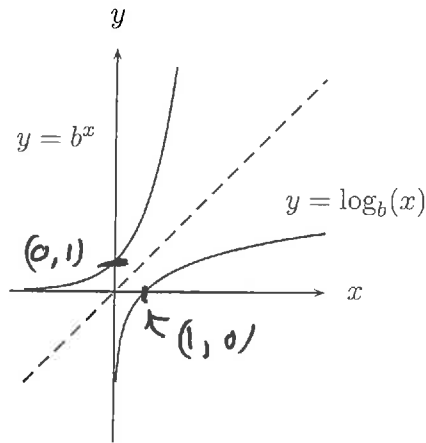
**In words:** The logarithm  $y = \log_a(x)$  is the power ( $y$ ) to which the base ( $a$ ) must be raised to get a given number ( $x$ )

$$\log_{10}(0.0000001) = \log_{10}(10^{-7}) = -7$$

Graphing logarithms:

Suppose we have the function  $y = f(x) = b^x$ .

Then the inverse function is  $x = b^y$  OR  $y = f^{-1}(x) = \log_b(x)$ . So the logarithm is the inverse of the exponential function.



$$\log_b(1) = ?$$

$$b^0 = 1$$

Can't use negative numbers.

Can't use zero.

2  
4  
8  
16  
32  
64

Since the two functions are inverses of each other then

$$b^{\log_b(x)} = x$$

$$\log_{10}(-10) = \text{?}$$

AND

$$\log_b(b^x) = x$$

$$10^y = -10$$

No answer.

Example 6.1.5. Simplify  $\log_2(64)$

$$\log_2(64) = \log_2(2^6) = \boxed{6}$$

$$\log_{10}(0) = \text{?}$$

$$10^y = 0$$

No answer.

## Properties of Logarithms

Let  $b$  be a positive real number with  $b \neq 1$ , and let  $x$  be any real number. Then:

1.  $\log_b(1) = 0$      i.e.  $b^0 = 1$
2.  $\log_b(b) = 1$      i.e.  $b^1 = b$
3.  $\log_b(b^x) = x$      i.e.  $b^x = b^x$
4.  $b^{\log_b(x)} = x$  if  $x \geq 0$

## The natural logarithm

This is the same as before but now we use base  $e$  where  $e$  is the number we found in section 6.1.1. Since the log base  $e$  shows up so often we call it the **natural log**.

$$\log_e(x) = \ln(x)$$

We also use log base 10 very often so we abbreviate that as

$$\log_{10}(x) = \log(x).$$

Your calculator follows the same convention.

**Example 6.1.6.** Evaluate the following logarithms without a calculator

$$1. \log(1000) = x \quad \Rightarrow \quad 10^x = 1000$$

$$\boxed{x=3}$$

$$2. \log_9(243) = x \quad 9^x = 243 \quad \Rightarrow \quad (3^2)^x = 3^5$$

$$3. \log_b(b^{-3}) = x = \boxed{-3}$$

$$b^x = b^{-3}$$

$$3^{2x} = 3^5$$

$$2x = 5$$

$$4. \ln(e^{-2}) = \log_e(e^{-2}) = -2$$

$$x = 5/2$$

**Example 6.1.7.** Evaluate with your calculator:

$$1. \log(345) = 2.537819095$$

$$2. \log(4/5) = -0.096910013$$

$$3. 3\ln(1 + \sqrt{3}) = 3.015157616$$

**Example 6.1.8.** More problems

1. Rewrite  $\log_{64} 8 = \frac{1}{2}$  in exponential form.

$$\boxed{64^{1/2} = 8}$$

2. Rewrite  $4^{-2} = \frac{1}{16}$  in logarithmic form.

$$\boxed{\log_4 \frac{1}{16} = -2}$$

3. Evaluate  $\log_8 2$ .

$$\boxed{= \frac{1}{3}}$$

4. Use the properties to evaluate  $\ln\left(\frac{1}{e^{21}}\right)$

$$= \ln(e^{-21})$$

$$\boxed{= -21}$$

5. Find the domain of  $g(x) = \ln(13 - x)$

6. Use the properties to evaluate  $-73 \ln(e)$

$$\boxed{= -73}$$

7. Use the properties to evaluate  $e^{\ln(55)}$

$$\boxed{= 55}$$

→ 5.  $13 - x > 0$

$$\boxed{13 > x}$$

$$8 = 2^3$$

$$8^x = 2 \Rightarrow 2^{3x} = 2^1$$

$$3x = 1$$

$$x = \frac{1}{3}$$

$$\log_e(e) = 1$$



## 6.2 Properties of Logarithms

### Change of Base Formula

Let  $a, b, x$  be positive real numbers with  $a \neq 1, b \neq 1$ . Then

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \quad (\text{For any } b)$$

For the calculator you can use either base 10 or base  $e$ .

$$\log_a(x) = \frac{\log(x)}{\log(a)} \quad \text{OR} \quad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

**Example 6.2.1.** Evaluate on a calculator using both common and natural logs.

$$\log_7(4) = \frac{\log 4}{\log 7} = 0.7124 = \frac{\log_7(4)}{\log_7(7)} = \frac{\ln 4}{\ln 7}$$

### Important Properties of Logarithms

Let  $b, M, N$  be positive real numbers with  $b \neq 1$ , and let  $p$  be any real number. Then:

1.  $\log_b(MN) = \log_b(M) + \log_b(N)$
2.  $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$
3.  $\log_b(M^p) = p \log_b(M)$
4.  $\log_b(M) = \log_b(N) \iff M = N$

**Example 6.2.2.** Use the properties of logs to rewrite the following expressions.

$$1. \log_{10}(10z) = \log 10 + \log z = 1 + \log z$$

$$2. \log_6\left(\frac{y}{2}\right) = \log_6(y) - \log_6(2)$$

$$3. \log_5(\sqrt[3]{5}) = \log_5(5^{1/3}) = \frac{1}{3} \log_5(5) = \frac{1}{3}$$

$$\sqrt[3]{5} = 5^{1/3}$$

$$\sqrt{5} = 5^{1/2} = \sqrt{5}$$

$$\log x - \log y + \log z$$

$$= \log\left(\frac{x}{y}\right) + \log z$$

$$= \log\left(\frac{x}{y} \cdot z\right) = \log\left(\frac{xz}{y}\right)$$

$$\log x - \log y + \log z$$

$$= \log x + \log y^{-1} + \log z$$

$$= \log(xz y^{-1})$$

$$f(z+5)$$

$$4. \ln\left(\frac{x^2-1}{x^3}\right) = \ln(x^2-1) - \ln(x^3)$$

$$= \ln[(x-1)(x+1)] - \ln(x^3) = \ln(x-1) + \ln(x+1) - 3\ln x$$

$$5. \log\left(\frac{\sqrt{xy^4}}{z^4}\right) = \log(x^{1/2}y^4) - \log(z^4)$$

$$= \log(x^{1/2}) + \log(y^4) - \log(z^4)$$

$$= \frac{1}{2}\log x + 4\log y - 4\log z$$

**Example 6.2.3.** Use the properties of logs to rewrite the following expressions as a single logarithm.

$$1. \ln(x-2) - \ln(x+2) = \ln\left(\frac{x-2}{x+2}\right)$$

$$2. \log_b w + \log_b x - \log_b y = \log_b\left(\frac{wx}{y}\right)$$

$$3. 4[\ln z + \ln(z+5)] - 2\ln(z-5) = 4\ln z + 4\ln(z+5) - 2\ln(z-5)$$

$$= \ln z^4 + \ln(z+5)^4 - \ln(z-5)^2$$

$$= \ln\left(\frac{z^4 \cdot (z+5)^4}{(z-5)^2}\right)$$

**Example 6.2.4.** Solve without a calculator:  $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$

$$= \log_4(64) = x$$

$$4^x = 64$$

$$x = 3$$

**Example 6.2.5.** More examples

1. Simplify by using common logarithms:  $\log_2(29) = \frac{\log 29}{\log 2}$

2. Simplify by using natural logarithms:  $\log_4 7 = \frac{\ln 7}{\ln 4}$

3. Write the expression in terms of  $\ln 5$  and  $\ln 3$ :  $\ln \frac{5}{81} = \ln \left( \frac{5}{3^4} \right)$   
 $= \ln(5) - \ln(3^4) = \boxed{\ln 5 - 4 \ln 3}$

4. Expand the logarithmic expression:  $\ln \frac{\sqrt{x-1}}{2}$   
 $= \ln \left( \frac{(x-1)^{1/2}}{2} \right) = \frac{1}{2} \ln(x-1) - \ln(2)$

5. Use the properties of logarithms to expand the expression:  $\log_6(12xy^3)$

$$\log_6(12) + \log_6(x) + 3 \log_6(y)$$

6. Write as a single logarithm:  $\frac{1}{3}[\log_6 x + \log_6(x+7)]$

$$= \frac{1}{3} (\log(x \cdot (x+7))) = \boxed{\log(x \cdot (x+7))^{1/3}}$$

7. Write as a single logarithm:  $\frac{1}{8}[\log_8(x-4) + 5 \log_8(x^2+9)]$

$$= \frac{1}{8} \log_8(x-4) + \frac{5}{8} \log_8(x^2+9)$$

$$= \log_8 \left[ (x-4)^{1/8} \cdot (x^2+9)^{5/8} \right]$$

$$\log 8$$

$$\log_8$$

$$\log_{-8} = \log_8$$

$$\log 18 = \log 8$$

## 6.3 Exponential Equations

Recall:

$$a^x = a^y \iff x = y$$

and

$$\log_a x = \log_a y \iff x = y$$

**Example 6.3.1.** Solve for  $x$  in the following equations:

1.  $64^{x+3} = 16$

$$(4^3)^{x+3} = 4^2$$

$$4^{3(x+3)} = 4^2$$

$$3(x+3) = 2$$

$$3x + 9 = 2$$

$$3x = -7$$

$$x = -\frac{7}{3}$$

2.  $3^x = 243$

$$3^x = 3^5$$

$$x = 5$$

3.  $\left(\frac{1}{4}\right)^x = 64$

$$4^{-x} = 4^3$$

$$-x = 3$$

$$x = -3$$

4.  $2^{x-3} = 32 = 2^5$

$$x - 3 = 5$$

$$x = 8$$

5.  $e^{9x} = 15$

$$\ln(e^{9x}) = \ln(15)$$

$$9x = \ln(15)$$

$$x = \frac{\ln(15)}{9}$$

6.  $2 + e^{x+2} = 32$

$$e^{x+2} = 30$$

$$\ln(e^{x+2}) = \ln(30)$$

$$x+2 = \ln(30)$$

$$x = \ln(30) - 2$$

$$2 + e^{\ln(30) - 2 + 2} = 32$$

Example 6.3.2. Solve for  $x$ :

1.  $e^{2x} = 50$

$$6^{5x} = 3000$$

$$\ln(6^{5x}) = \ln(3000)$$

2.  $6^{5x} = 3000$

$$5x \frac{\ln(6)}{\ln(6)} = \frac{\ln(3000)}{5 \ln(6)}$$

$$x = \frac{\ln(3000)}{5 \ln(6)} = 0.893687764$$

3.  $\frac{119}{e^{6x} - 14} = 7$

$$\ln(3000) / 5 / \ln 6 = \ln(3000) / (5 \ln 6)$$

$$\frac{\ln(3000)}{5 \ln 6}$$

## 6.4 Logarithmic Equations

**Example 6.4.1.** Solve for  $x$  in the following equations:

1.  $\ln x - \ln 5 = 0$

$$\ln x = \ln 5$$

$$\boxed{x = 5}$$

2.  $\log_x 625 = 4 \Rightarrow x^4 = 625 = 5^4$

$$\boxed{x = 5}$$

3.  $\ln(2x - 1) = 0 \Rightarrow e^0 = 2x - 1$

$$1 = 2x - 1$$

$$\boxed{x = 1}$$

4.  $5 + 5 \ln x = 30$

$$\begin{array}{r} 5 + 5 \ln x = 30 \\ -5 \quad -5 \end{array}$$

$$5 \ln x = 25$$

$$\ln x = 5 \Rightarrow$$

$$\boxed{e^5 = x}$$

← get  $\ln x$  alone.

5.  $\log 2x - \log 8x^2 = 3$

$$\log\left(\frac{2x}{8x^2}\right) = 3 \Rightarrow \log\left(\frac{1}{4x}\right) = 3$$

$$10^3 = \frac{1}{4x}$$

$$\boxed{x = \frac{1}{4000}}$$

**Example 6.4.2.** Simplify the following:

1.  $\log_6 6^{2x-1} = y = \boxed{2x-1}$

$$6^y = 6^{2x-1}$$

2.  $\ln e^{x^4} =$

$$\boxed{x^4}$$

Example 6.4.3. Solve for  $x$ :

$$1. \ln(4x) = 1 \Rightarrow \frac{e^1}{4} = \frac{4x}{4}$$

$$\boxed{\frac{e}{4} = x}$$

$$2. \log_6(6x) + \log_6(x+5) = 2$$

← write as one log using

$$\log(MN) = \log(M) + \log(N)$$

$$\log_6 [6x(x+5)] = 2$$

$$6^2 = 6x(x+5)$$

$$36 = 6x^2 + 30x$$

$$0 = 6x^2 + 30x - 36$$

$$0 = 6(x^2 + 5x - 6)$$

$$0 = 6(x+6)(x-1)$$

$$\cancel{x = -6}$$

$$\& \boxed{x = 1}$$

Check  
Domain.  
Doesn't  
work  
in  
original.

$$\log_6(-6x) = 2$$

$$6^2 = -6x$$

$$\boxed{-6 = x}$$

yes



$$\log 30 - \log 10 = \log 3$$

$$\underline{\log(3 \cdot 10)} - \log 10$$

$$\log 3 + \log 10 - \log 10 = \log 3 \checkmark$$

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$$\#7 \quad \underline{\log_2(\log_2 x)} = \underline{3}$$

$$8 = 2^3 = \log_2 x$$

$$2^8 = x$$

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$$\log_6 x^2 - \log_6(x+3) = 9$$

$$\log_6 \left( \frac{x^2}{x+3} \right) = 9$$

$$(x+3)(6^9) = \left( \frac{x^2}{x+3} \right)(x+3) \quad \frac{-6^9 \pm \sqrt{(6^9)^2 - 4(-1)(3 \cdot 6^9)}}{2(-1)}$$

$$-x^2 + 6^9 x + 3(6^9) = \cancel{x^2} 0$$

$$\log_3 x + \log_3 (x+5) = 1$$

$$\log_3 (x(x+5)) = 1$$

$$3^1 = x(x+5) \leftarrow \text{quadratic}$$

$$3 = x^2 + 5x$$

$$0 = x^2 + 5x - 3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 - 4(1)(-3)}}{2(1)}$$

$$= \frac{-5 + \sqrt{37}}{2}$$

## 6.5 Exponential and Logarithmic Models

**Example 6.5.1** Find the time required to double an investment of  $P_0$  dollars if the interest rate is 6% compounded continuously.

The formula for interest compounded continuously is

$$P_t = P_0 e^{rt}$$

$P_t$  = amount in year  $t$

$P_0$  = Initial Investment

$r$  = interest rate

$t$  = years.

Solve for  $t$ .

To double the investment we want to find the value of  $t$  when  $P_t = 2P_0$ .

$$\frac{2P_0}{P_0} = \frac{P_0 e^{(0.06)t}}{P_0}$$

$$\ln(2) = \ln[e^{(0.06)t}]$$

$$\frac{\ln 2}{0.06} = \frac{(0.06)t}{0.06}$$

$$t = 11.55 \text{ years}$$

**Example 6.5.2.** The population of a city is

$$P = 240360e^{0.012t}$$

where  $t = 0$  represents the year 2000. According to this model when will the population reach 275000?

What is growth rate?  $r = 0.012$  or 1.2%

How many people in the city in 2000? 240,360

$$\frac{275000}{240360} = \frac{240360 e^{0.012t}}{240360}$$

$$\frac{\ln\left(\frac{275000}{240360}\right)}{0.012} = \ln(e^{0.012t}) = \frac{0.012t}{0.012}$$

$$t = 11.22 \text{ years.}$$

$$2000 + 11.22 = \boxed{2011} \leftarrow \text{year}$$

**Example 6.5.3.** The number of bacteria  $N$  in a culture is modeled by

$$N = 250e^{kt}$$

where  $t$  is the time in hours. If  $N = 280$  when  $t = 10$ , estimate the time required to double the population.

step 1: Find  $k$  by solving the formula  $280 = 250e^{k(10)}$ .

step 2: Find  $t$  for  $N = 2(250) = 500$ . (double the original population)

$$\frac{280}{250} = \frac{250 e^{10k}}{250}$$

$$\ln\left(\frac{28}{25}\right) = \ln(e^{10k})$$

$$\ln\left(\frac{28}{25}\right) = 10k$$

$$k = 0.011332869$$

$$\frac{500}{250} = \frac{250 e^{kt}}{250}$$

$$\ln(2) = \ln(e^{kt})$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}$$

$t = 61$   
hours.

**Example 6.5.4.** Carbon 14 ( $^{14}\text{C}$ ) has a half life of 5730 years. (Half life is the amount of time for half the original material to decay.) Carbon 14 dating assumes that the carbon dioxide today has the same amount of radioactive material as it did centuries ago. If this is true, the amount of  $^{14}\text{C}$  absorbed by a tree centuries ago should be the same as a tree growing today. A piece of ancient coal has 15% as much  $^{14}\text{C}$  as a piece of modern coal. How long ago was the tree burned to make the ancient coal?

Decay Model:

$$A = A_0 e^{-kt}$$

where  $A_0$  is the original amount of material,  $k$  is the decay constant and  $A$  is the amount of material left after  $t$  years.

Step 1: Find  $k$  using the half life.  $\frac{1}{2}A_0 = A_0 e^{-k5730}$

Step 2: Find  $t$  for  $0.15A_0 = A_0 e^{-kt}$ .

Find t

$$\frac{1}{2} = e^{-k(5730)}$$

$$\ln\left(\frac{1}{2}\right) = \frac{-5730k}{-5730}$$

$$\ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2)$$

$$= -\ln 2$$

$$k = 0.000120968$$

$$0.15A_0 = A_0 e^{-kt}$$

$$\ln(0.15) = \frac{-kt}{-k}$$

$$t = 15683 \text{ years old}$$

1.209 ~~E-4~~  $\times 10^{-4}$