# 5 Further Topics in Functions

## 5.1 Function Composition

### Composition of Function

The **composition** of a function $f$ with a function $g$ is

$$(f \circ g)(x) = f(g(x)).$$ $(g \circ f)(x) = g(f(x))$

The domain of $(f \circ g)$ is the set of all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

**Example 5.1.1.** Suppose $f(x) = x^3 + 2x + 1$ and $g(x) = x - 1$ then find

a) $(f \circ g)(x) = f(g(x)) = f(x-1) = (x-1)^3 + 2(x-1) + 1$
$= x^3 - 3x^2 + 3x - 1 + 2x - 2 + 1 = \boxed{x^3 - 3x^2 + 5x - 2}$

b) $(g \circ f)(x) = g(f(x)) = g(x^3 + 2x + 1)$
$= x^3 + 2x + 1 - 1 = \boxed{x^3 + 2x}$

c) $(f \circ f)(x) = f(f(x)) = f(x^3 + 2x + 1)$
$\boxed{= (x^3 + 2x + 1)^3 + 2(x^3 + 2x + 1) + 1}$

**Example 5.1.2.** Find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$ and (c) the domain of each for $f(x) = \sqrt{x-4}$ and $g(x) = x^2$.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{x^2 - 4}$

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x-4}) = (\sqrt{x-4})^2 = x - 4$

Domain

**Example 5.1.3.** Find functions $f$ and $g$ such that $h(x) = \sqrt[3]{x^2 - 4} = (f \circ g)(x)$.

$f(x) = \sqrt[3]{x}$ $\quad$ $f(g(x)) = f(x^2 - 4)$
$g(x) = x^2 - 4$ $\quad$ $= \sqrt[3]{x^2 - 4}$

or $f(x) = \sqrt[3]{x-4}$ $\quad$ $f(g(x)) = f(x^2)$
$g(x) = x^2$ $\quad$ $= \sqrt[3]{x^2 - 4}$

Ex 5.1.2 Domain $f(x^2)$

Domain of $(f \circ g)(x) = \sqrt{x^2 - 4}$

$x^2 - 4 \geq 0$

$(x-2)(x+2) \geq 0$

Zeros $x = -2, 2$

| $-2$ | | $2$ |
|---|---|---|
| $x = -3$ | $x = 0$ | $x = 3$ |
| $+$ | $-$ | $+$ |

$(-\infty, -2]$ $[2, \infty)$

$x \leq -2$ $x \geq 2$

Be careful

$\sqrt{x^2} \geq \pm\sqrt{4}$

$x \geq \sqrt{4}$

or

$x \leq -\sqrt{4}$

Domain of $(g \circ f)(x) = g(f(x)) = x - 4$

$g(\underbrace{\sqrt{x-4}})$

can only use things that would work in $f(x)$

$x - 4 \geq 0$

Domain $\boxed{x \geq 4}$

or $f(x) = \sqrt[3]{x^2-4}$

$g(x) = x$

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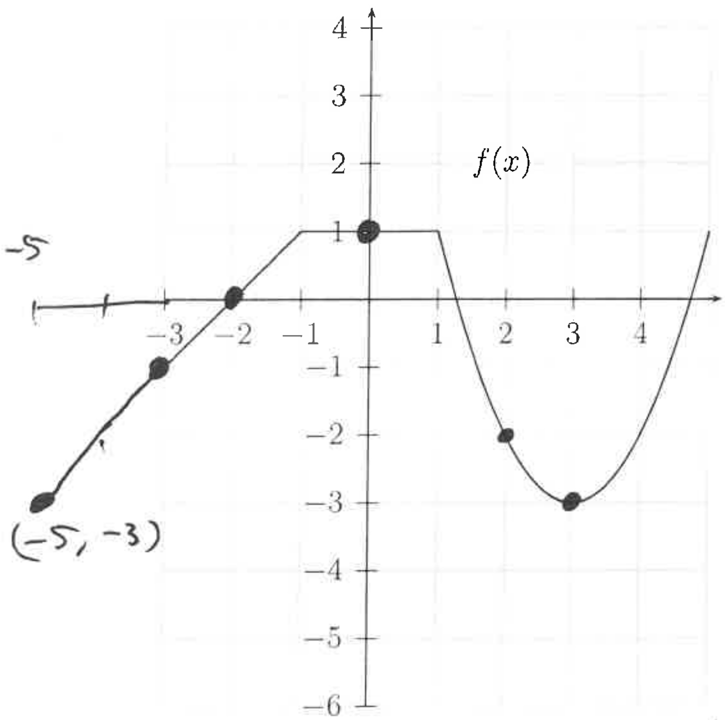
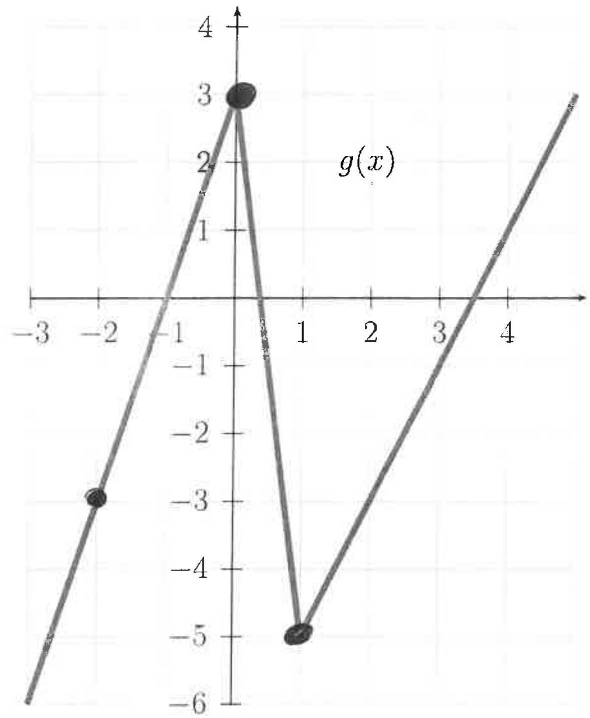
or $\cancel{g}\, f(x) = x$

$g(x) = \sqrt[3]{x^2-4}$

**Example 5.1.4.** Use the functions below to find the compositions.




1. $f(g(1)) = f(-5) = \boxed{-3}$
2. $g(f(2)) = g(-2) = \boxed{-3}$
3. $f(g(0)) = f(3) = \boxed{-3}$
4. $g(f(0)) = g(1) = \boxed{-5}$
5. $f(g(-2)) = f(-3) = \boxed{-1}$
6. $g(f(-2)) = g(0) = \boxed{3}$

§5.2

What is the inverse of 7?

Need to know operation!

For 7 there are two possible operations

addition

& multiplication.

<u>Identity element.</u>

Identity does not change what it is operating on.

Additive identity = 0

$0+7=7$

multiplicative identity = 1

$1(7)=7$

The inverse of a number is another number that when combined with the first gives the identity.

Additive inverse of 7 = $-7$ b/c $7-7=0$

multiplicative inverse of 7 = $\frac{1}{7}$ b/c $7(\frac{1}{7})=1$

For functions we need

Operation: composition of functions.

Identity: $\underbrace{I(x) = x}$

Does not change the value of what goes in.

§5.2

What is the inverse of 7?

Need to know operation!

For 7 there are two possible operations

addition

& multiplication.

## Identity element.

Identity does not change what it is operating on.

Additive identity $= 0$

$$0 + 7 = 7$$

multiplicative identity $= 1$

$$1(7) = 7$$

The inverse of a number is another number that when combined with the first gives the identity.

Additive inverse of 7 $= -7$ b/c $7 - 7 = 0$

multiplicative inverse of 7 $= \frac{1}{7}$ b/c $7\left(\frac{1}{7}\right) = 1$

For functions we need

Operation: composition of functions.

Identity: $\underbrace{I(x) = x}_{}$

Does not change the value of what goes in.

## 5.2 Inverse Functions

### One to one functions

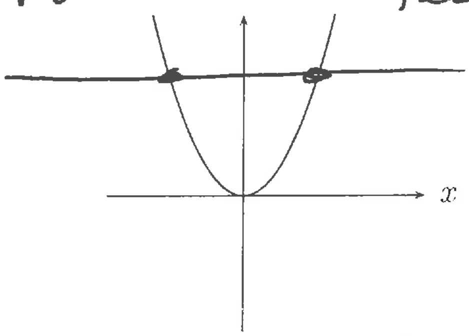
A function is said to be **one to one (1 - 1)** if no two ordered pairs have the same second component but different first component.

A function has one $y$ value for each $x$ value but those $y$ values can repeat. In a 1 - 1 function the $y$ values never repeat.

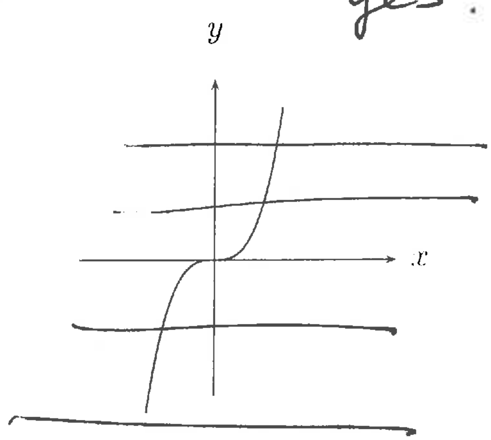
**Graphically:**
An equation must pass the Vertical Line Test to be a function.
A function must pass the Horizontal Line Test to be 1 - 1.

**Example 5.2.1.** $f(x) = x^2$ is not 1 - 1. $f(x) = x^3$ is 1 - 1.

Fails horizontal line test.




yes.

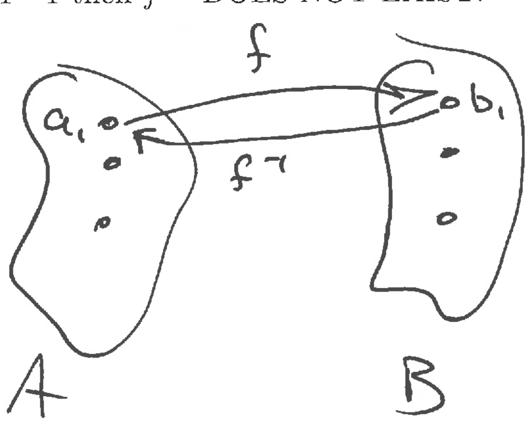



### Inverses

The **indentity function** is $f(x) = x$ or $y = x$. You get out what you put in. Given a function $f$ that is 1 - 1 then $f$ has an inverse $f^{-1}$ and

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

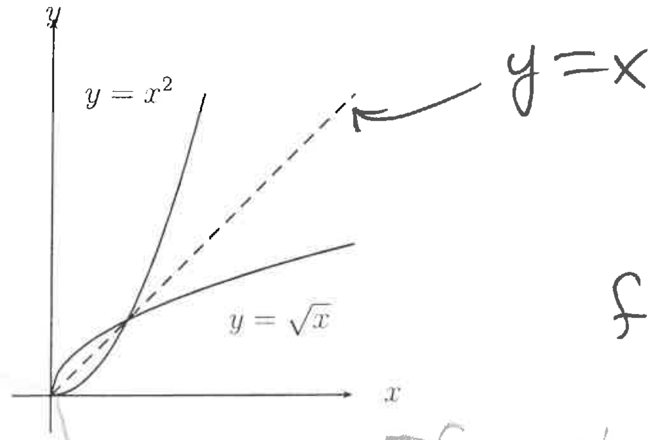
If $f$ is not 1 - 1 then $f^{-1}$ DOES NOT EXIST.




$f: a_1 \to b_1$

$f^{-1}: b_1 \to a_1$

**Graphically**



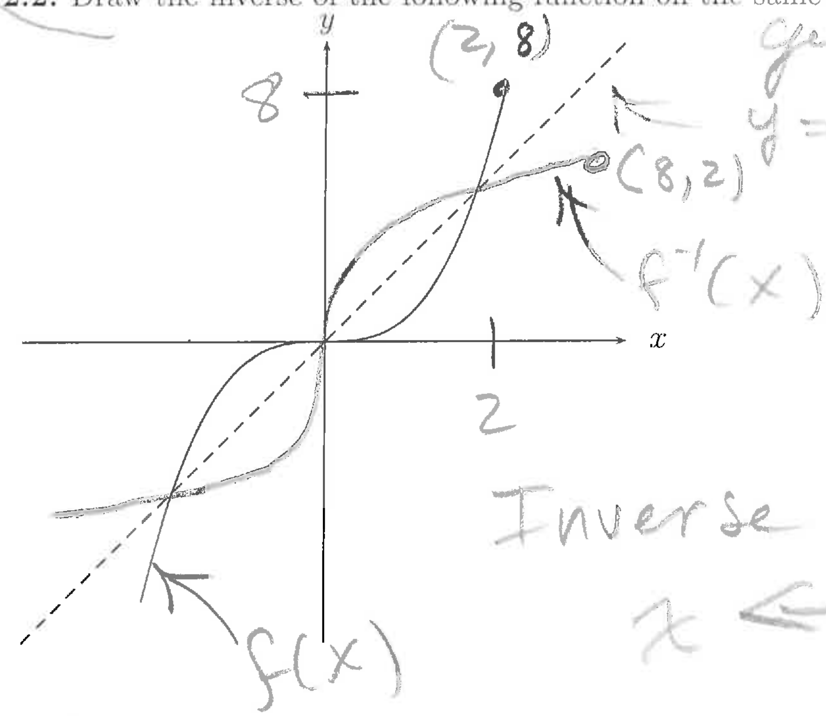

$f(x) = y = x^2 \quad x \geq 0$

$f^{-1}(x) = \sqrt{x} \quad x \geq 0$

$f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2$
$= x$

If not 1-1 then when you reflect over $y=x$ you do not get a function.

**Example 5.2.2.** Draw the inverse of the following function on the same set of axes.




Inverse reverses
$x \Longleftrightarrow y$

## Finding Inverses Algebraically

**Example 5.2.3.** Find the inverse of $f(x) = -x^2 + 4x$ on $x \leq 2$

**Step 1:** Solve for $x$.

$$\begin{aligned} y &= -x^2 + 4x \\ 4 - y &= x^2 - 4x + 4 \\ 4 - y &= (x-2)^2 \end{aligned}$$

**Step 2:** Check the domain.

$$x \leq 2 \quad \text{so use} \quad x = 2 - \sqrt{4-y}$$

**Step 3:** Switch $x$ and $y$.

$$y = 2 - \sqrt{4-x}$$

**Step 4:** Write

$$\boxed{f^{-1}(x) = 2 - \sqrt{4-x}.}$$

**Step 5:** Check that $f(f^{-1}(x)) = x$.

**Example 5.2.4.** Find the inverse of $f(x) = \sqrt{4-x^2}$. $0 \leq x \leq 2$

$$(y)^2 = \left(\sqrt{4-x^2}\right)^2 \leftarrow \text{solve for } x$$

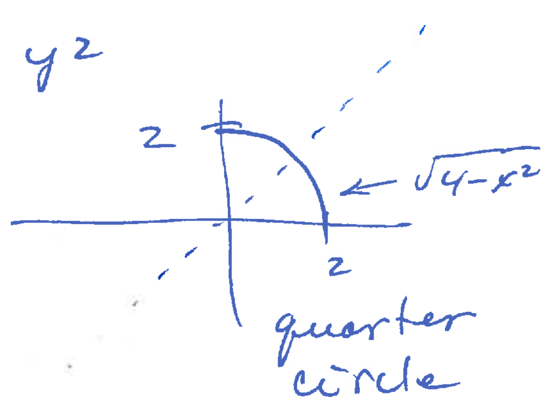
$$-y^2 + x^2 + y^2 = 4 - x^2 + x^2 - y^2$$

$$\sqrt{x^2} = \pm\sqrt{4-y^2}$$

$$x = \pm\sqrt{4-y^2}$$

$$x = +\sqrt{4-y^2} \leftarrow \text{Domain}$$

$$\boxed{f^{-1}(x) = y = \sqrt{4-x^2}}$$



$f(x) = -x^2 + 4x$ on $x \le 2$

Find $f^{-1}(x)$

Domain

$y = -x^2 + 4x$

solve for $x$

$x =$ stuff that is not $x$

$-y = -1(-x^2 + 4x)$

Need a 1 in front of $x^2$.

$+4 - y = x^2 \underbrace{-4}_{} x + 4$

(half of this)$^2$

$\pm\sqrt{4-y} = \sqrt{(x-2)^2}$

$+2 \pm \sqrt{4-y} = x - 2 + 2$

$2 \pm \sqrt{4-y} = x$

can only use one of these.
So check domain: $x \le 2$

use negative because

$2 - \sqrt{\text{something}} \le 2$