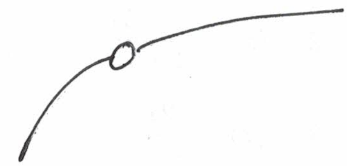
4 Rational Functions

4.1 Introduction to Rational Functions

$\frac{\text{zero}}{\text{zero}}$ is a hole.



Q: What is a rational function?

A: It is a function of the form:

$$f(x) = \frac{N(x)}{D(x)} \qquad D(x) \neq 0.$$

Rational functions can have **Vertical and Horizontal Asymptotes**

A **Vertical Asymptote** describes the behavior of a function near a discontinuity. They occur at any x - value where the numerator IS NOT equal to zero but the denominator IS equal to zero.

Example 4.1.1. Find vertical asymptotes for

$$f(x) = \frac{1}{x} \text{ and } f(x) = \frac{1}{x-3} \text{ and } f(x) = \frac{1-3x}{x(x-3)}.$$

$f(x) = \frac{1}{x}$

$\boxed{x = 0 \text{ V.A.}}$

$f(x) = \frac{1}{x-3}$

$x - 3 = 0$

$\boxed{x = 3 \text{ V.A.}}$

$f(x) = \frac{1-3x}{x(x-3)}$

$x(x-3) = 0$

$\boxed{x = 0}$ or $x - 3 = 0$

V.A. $\boxed{x = 3}$

A **Horizontal Asymptote** describes the behavior of a function as x gets very large. (ie. What happens to y as x goes to ∞?)

> **Horzontal Asymptotes**
>
> Let f be the rational function given by
>
> $$f(x) = \frac{N(x)}{D(x)} = \frac{\boxed{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{\boxed{b_m x^m} + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$
>
> where $N(x)$ and $D(x)$ have no common factors. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $n(x)$ and $D(x)$.
>
> 1. If $n < m$, then the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
> 2. If $n = m$ then the graph of f has the line $y = \frac{a_n}{b_m}$ as a horzontal asymptote.
> 3. if $n > m$ then the graph of f has no horizontal asymptote.

#~~10~~8

$x=3$ mult 2

$x=0$, $x=-3$

$(5,112)$

$P(x) = A(x-3)^2 x(x+3)$

$P(5) = 112$

$x=3$	$(x-3)$	Factor
$x=-3$	$(x+3)$	Factor
$x=0$	(x)	Factor

$$\frac{12x^3-67x^2+80x-16}{x-4} = \frac{(x-4)\,q(x)}{x-4}$$

$$\begin{array}{r|rrrr} 4 & 12 & -67 & 80 & -16 \\ & & 48 & -76 & 16 \\ \hline & 12 & -19 & 4 & 0 \end{array}$$

$0 = 12x^2 - 19x + 4$ ⇐ quadratic formula

#31

$$\frac{2x^3 - 12x^2 + 7x - 29}{2x^2+5} = x - 6 + \frac{2x+1}{2x^2+5}$$

$$\begin{array}{r@{}l} & \quad\;\; x - 6 \\ 2x^2+5 \,\big) & \, 2x^3 - 12x^2 + 7x - 29 \\ & -(2x^3 + 0x^2 + 5x) \\ \hline & \quad -12x^2 + 2x - 29 \\ & -(-12x^2 + 0x - 30) \\ \hline & \qquad\quad 2x + 1 \end{array}$$

#9 degree 3

mult 2 $x = 4$

mult 1 $x = -5$

When $x = 0$

$y = -8$ $\longrightarrow$ $y = -8$ y-int

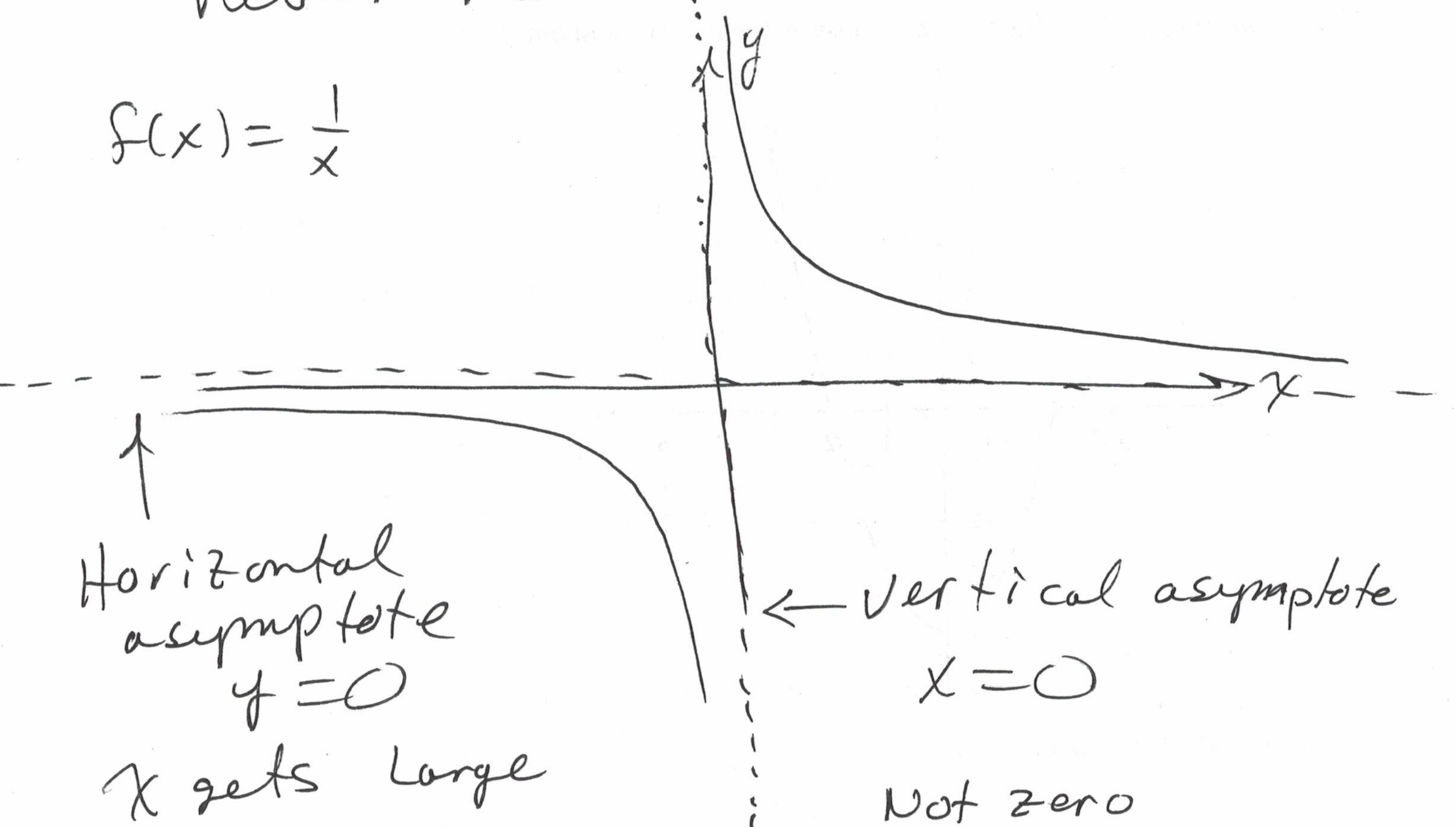
$$P(x) = A(x-4)^2(x+5)$$

$$A(-4)^2(5) = -8$$

$$A = \frac{-\cancel{8}\;1}{2\,\cancel{16}(5)} = -\frac{1}{10}$$

__Asymptote:__ A line that the graph gets close to but never reaches.

$f(x) = \frac{1}{x}$

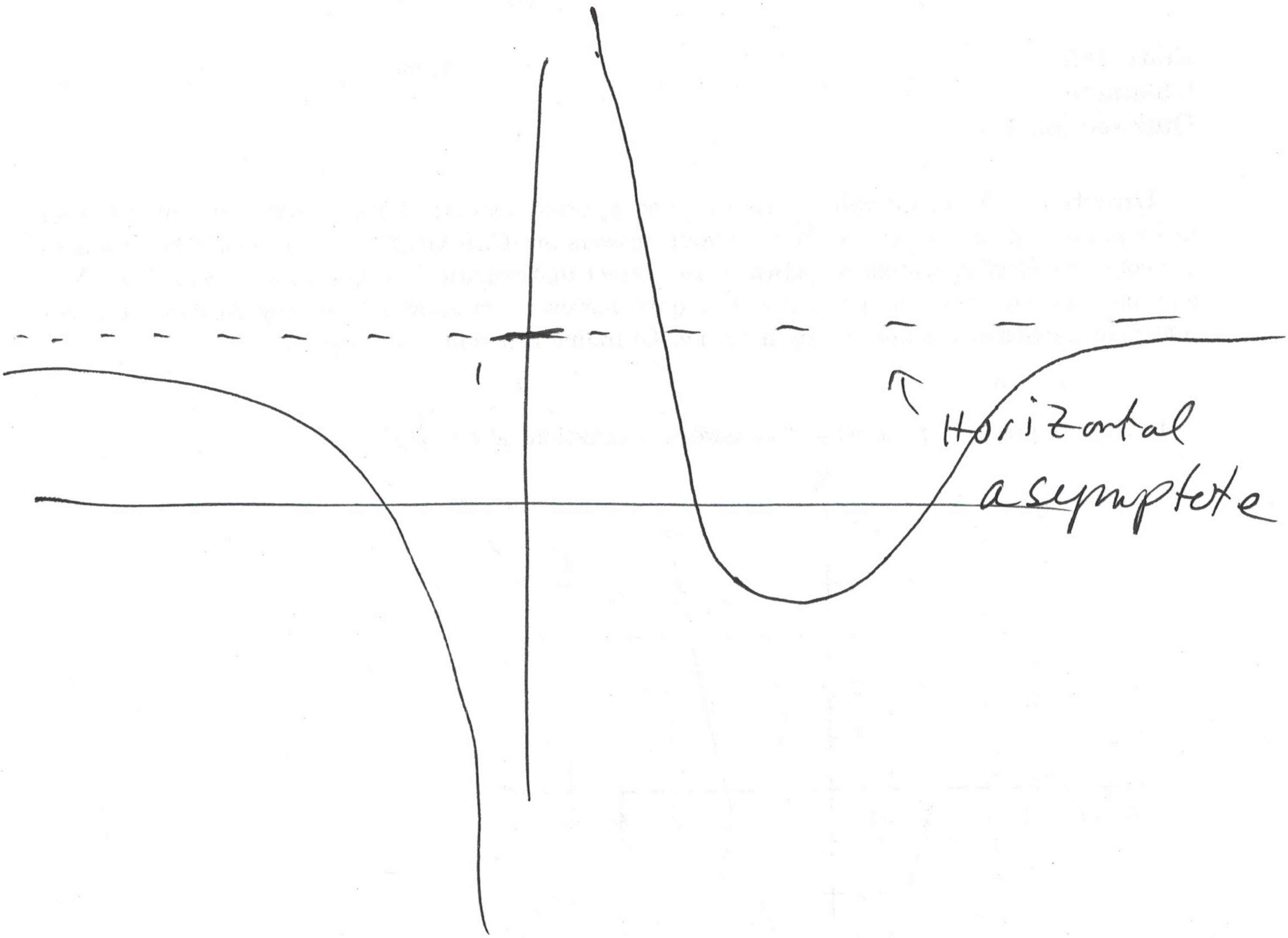


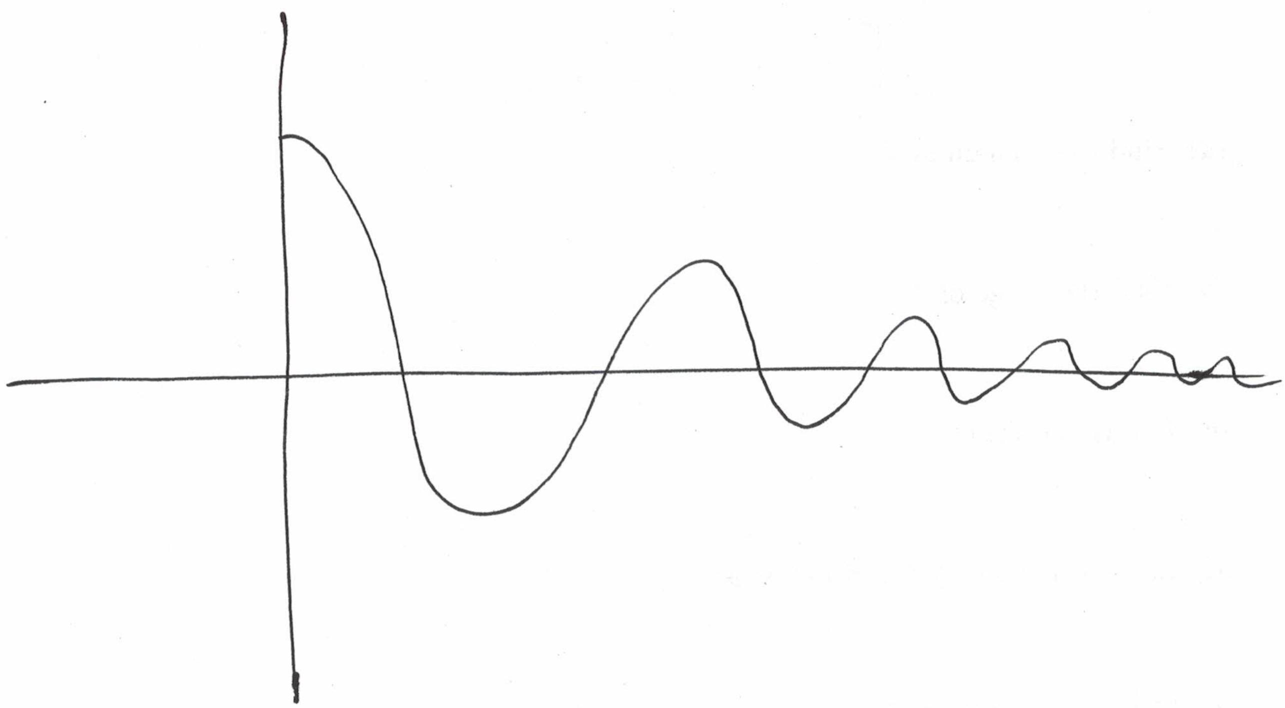
Horizontal asymptote
$y = 0$

x gets Large
$x \to \infty$
or $x \to -\infty$

vertical asymptote
$x = 0$

$\frac{\text{Not zero}}{\text{zero}}$


Horizontal
asymptote



Horizontal Asymptote

$$f(x) = \frac{3x^2 + 4x - 122}{5x^2 + 3x}$$

What happens to $y = f(x)$ as $x \to \infty$

$$\frac{1}{x} \to 0 \quad \text{as} \quad x \to \infty$$

Divide every term by the largest power of x in the denominator.

$$f(x) = \frac{\frac{3x^2}{x^2} + \frac{4x}{x^2} - \frac{122}{x^2}}{\frac{5x^2}{x^2} + \frac{3x}{x^2}}$$

$$= \left.\frac{3 + \cancel{\frac{4}{x}} \to 0 - \cancel{\frac{122}{x^2}} \to 0}{5 + \cancel{\frac{3}{x}} \to 0}\right\} \text{let } x \to \infty$$

$$\boxed{y = \frac{3}{5} \quad \text{Horizontal asymptote}}$$

#1

$$f(x) = \frac{4}{(x-2)^2}$$

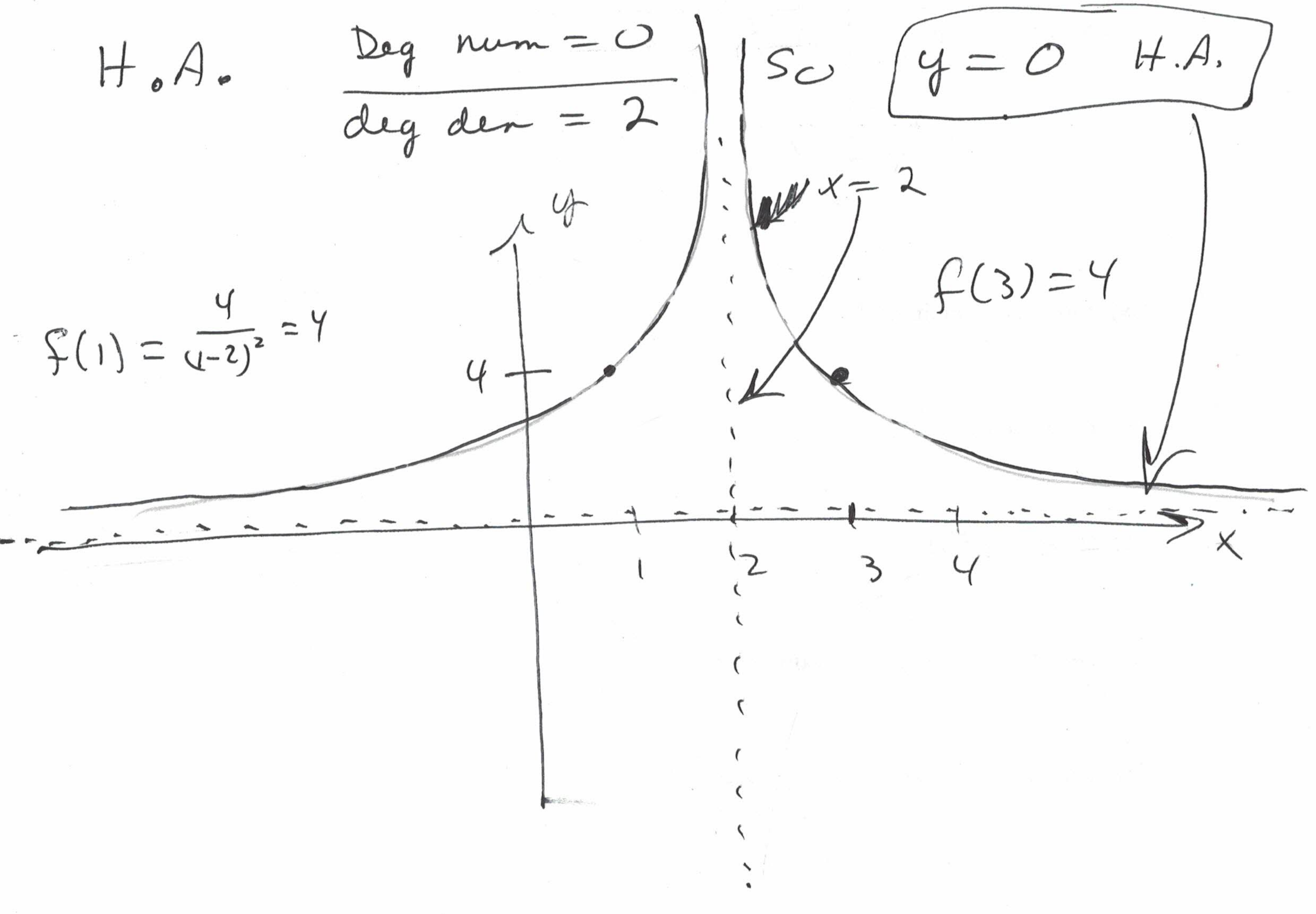
Domain & graph & Asymptotes.

Domain $(x-2)^2 \neq 0$

$$\boxed{x \neq 2}$$

V. A. $\boxed{x = 2}$

H. A. $\frac{\text{Deg num} = 0}{\text{deg den} = 2}$ so $\boxed{y = 0 \quad H.A.}$



#2) $f(x) = \dfrac{1-5x}{1+2x}$

Asymptotes, domain & graph with (x-int)

Domain $1+2x \neq 0$

$$\boxed{x \neq -\tfrac{1}{2}}$$

V.A. $\boxed{x = -\tfrac{1}{2}}$

H.A. $\boxed{y = -\tfrac{5}{2}}$ $\qquad \dfrac{\text{deg num} = 1}{\text{deg den} = 1}$

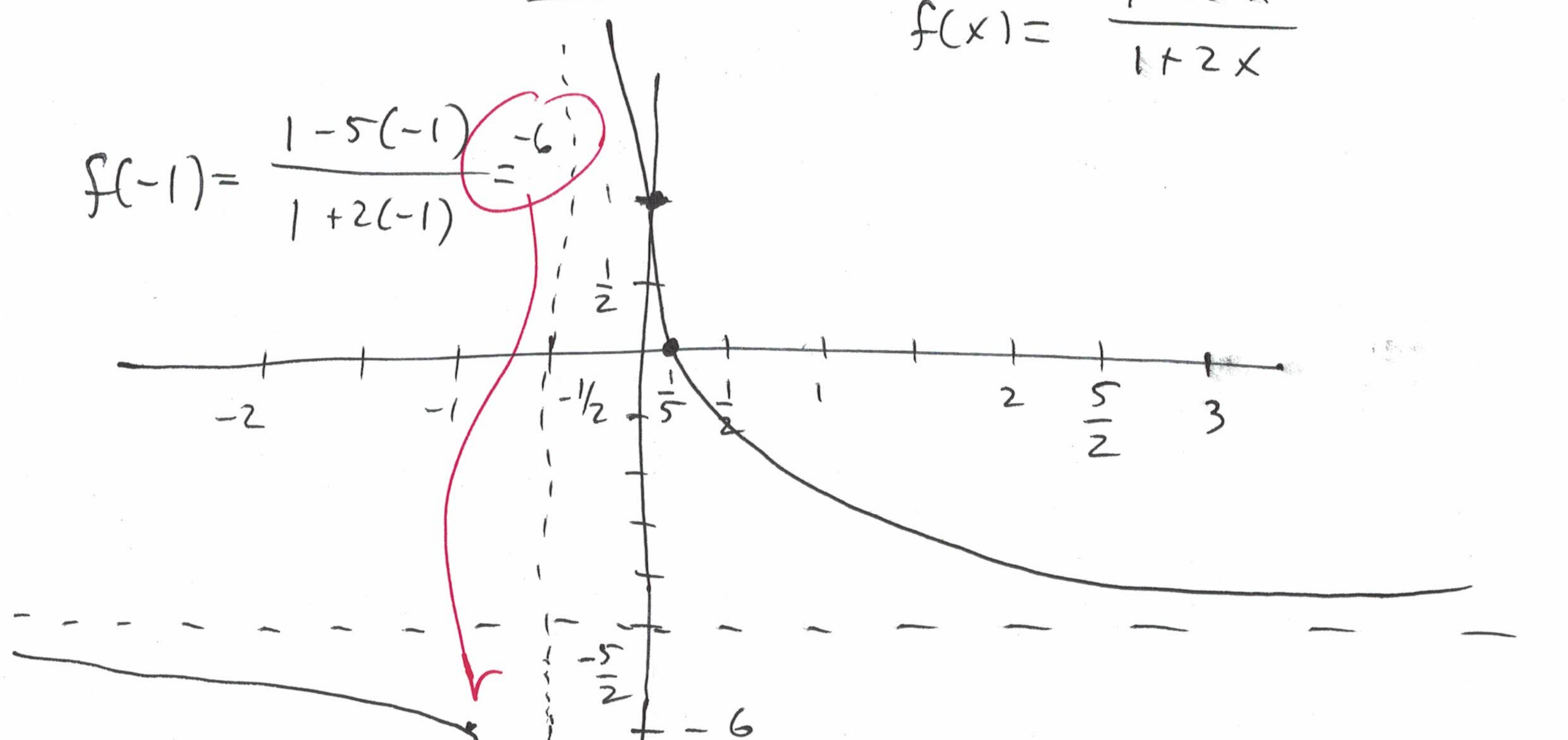
x-int: $1-5x=0$

$$\boxed{x = \tfrac{1}{5}}$$

y-int: $\boxed{y=1}$ $\qquad f(0)=1$

$f(x) = \dfrac{1-5x}{1+2x}$

$f(-1) = \dfrac{1-5(-1)}{1+2(-1)} = -6$



Ex

$$f(x) = \frac{3x^2 + 17x + 10}{2x^2 + 11x + 12}$$

y - int: $f(0) = \frac{10}{12} = \frac{5}{6}$

x - int:

Vertical asymptotes

Horizontal asymptote $\boxed{y = \frac{3}{2}}$

top = 0 x - int

bottom = 0 Vertical asymptotes.

$10 = 5 \cdot 2$
$= 10 \cdot 1$

$3x^2 + 17x + 10 = 0$

$(3x + 2)(x + 5) = 0$

x - int $3x + 2 = 0$ or $x + 5 = 0$

$\boxed{x = \frac{-2}{3}}$ $\boxed{x = -5}$

$12 = 6 \cdot 2$
$= 3 \cdot 4$
$= 12 \cdot 1$

$2x^2 + 11x + 12 = 0$

$(2x + 3)(x + 4) = 0$

V.A. $2x + 3 = 0$ $x + 4 = 0$

$\boxed{x = -\frac{3}{2} \qquad x = -4}$

Ex: Vertical asymptotes at

$x = 1 \quad x = -3$

x-intercepts $x = -6$, $x = -5$

y-intercept at $y = 9$

$$f(x) = A \frac{\text{x-intercepts}}{\text{vertical asymptotes}}$$

$$f(x) = A \frac{(x+6)(x+5)}{(x-1)(x+3)}$$

$$f(0) = 9 = A\left(\frac{6 \cdot 5}{(-1)(3)}\right)$$

$$\frac{9}{-10} = \frac{A(\cancel{-10})}{\cancel{-10}} \; 1$$

$$\boxed{f(x) = \frac{-9(x+6)(x+5)}{10(x-1)(x+3)}}$$

§4.3 Equations with rational functions

Word problem

suppose f varies inversely with y and when $f = 32$ $y = 5$.
What is the value of f when $y = 8$?

constant

f varies directly with y $\quad f = ky$

f varies inversely with y $\quad f = \frac{k}{y}$

$$32 = \frac{k}{5} \qquad f = \frac{160}{y}$$

$$k = 160$$

$$y = 8 \qquad f = \frac{160}{8} = 20$$

$\cancel{\frac{32}{5} = \frac{x}{8}}$

solve the equation

$$\frac{11}{x} = \frac{10}{3x} + 9$$

multiply by common denominator $3x$

$$\frac{3x}{1}\left(\frac{11}{x}\right) = \frac{3x}{1}\left(\frac{10}{3x} + 9\right)$$

$$\frac{33x}{x} = \frac{30x}{3x} + 27x$$

$$33 = 10 + 27x$$

$$\boxed{x = \frac{23}{27}}$$

$$\frac{x}{4x-16} - 6 = \frac{1}{x-4}$$

$$\frac{4(x-4)}{1}\left(\frac{x}{4(x-4)} - 6\right) = \left(\frac{1}{x-4}\right)\frac{4(x-4)}{1}$$

$$\frac{\cancel{4}\,x\,\cancel{(x-4)}}{\cancel{4}\cancel{(x-4)}} - 6(4)(x-4) = \frac{4\cancel{(x-4)}}{\cancel{(x-4)}}$$

$$x \underbrace{-24}(x-4) = 4$$

$$x - 24x + 96 = 4$$

$$\frac{-23x}{-23} = \frac{-92}{-23}$$

$$\cancel{x = 4}$$

Not in domain of