Ch 3

Polynomials

Generic polynomial of degree n.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x^1 + a_0$$

$a_n, a_{n-1}, \ldots, a_1, a_0$ coefficients

leading term

leading term: term with largest degree

Ex. $f(x) = x^2 \boxed{-3x^4}$

leading term $= -3x^4$

degree: exponent on the leading term.

Ex. $f(x)$ has degree 4

leading coefficient: number in front of the variable of the leading term.

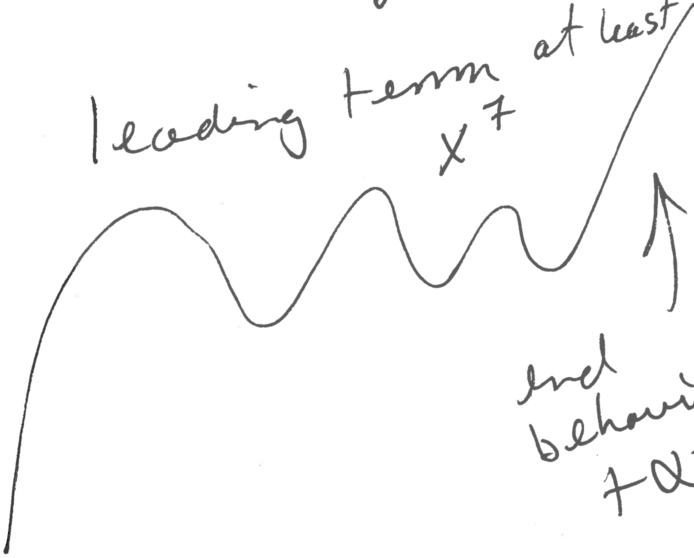
Ex: the leading coefficient of $f(x)$ is -3

Constant term: a_0 for $f(x)$ $a_0 = 0$

End behavior

two general forms

leading term at least x^7



end behavior $+\infty$

leading term at least x^{10}

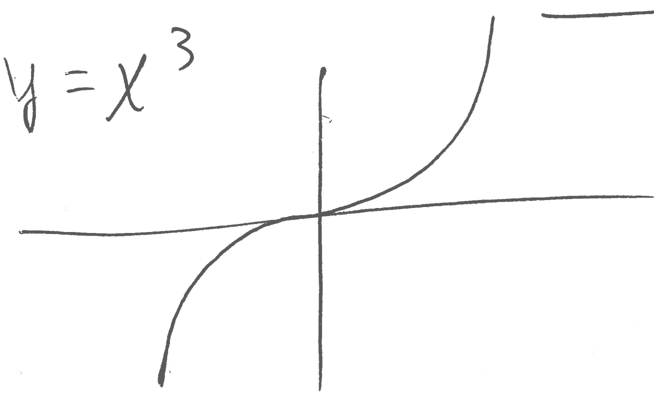
~~odd leading~~
odd degree

even degree

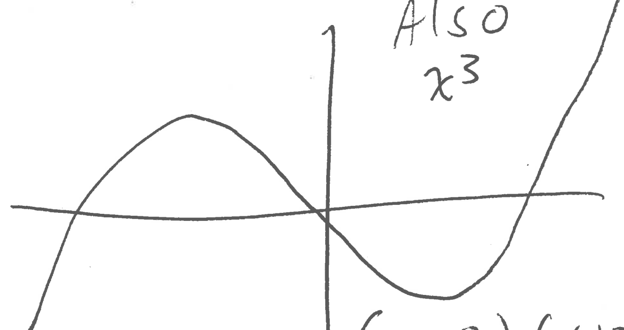
end behavior is $-\infty$

Number of direction changes is at **most** $n-1$

$y = x^3$



Also x^3



$(x-2)(x+2)(x)$

Zeros, roots, facters, x-intercepts

(or root)
$x = 5$ is a zero of
$f(x)$ iff $f(5) = 0$

x-intercept where crosses x-axis
so if $x = 5$ is an x-intercept
then $f(5) = 0$.

If $x = 5$ is a zero of
$f(x)$ then $(x-5)$ is
a factor of $f(x)$.

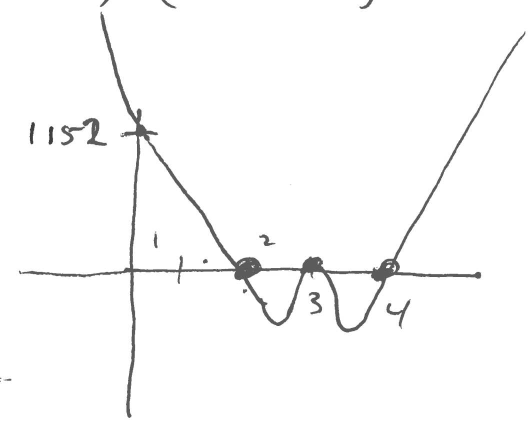
If
$f(5) = 0$ ~~(x-5) q(x)~~)

then $f(x) = (x-5)\, q(x)$

Multiplicity of zeros

Multiplicity is the number of times the zero shows up in the factorization.

Ex! $f(x) = (x-2)(x-3)^2(x-4)^3$



$f(x)$ has degree 6

$f(x)$ has 3 zeros

$x = 2$ mult. 1

$x = 3$ mult. 2

$x = 4$ mult. 3

leading term $= x^6$

$f(x) = (x-2)(x-3)(x-3)(x-4)(x-4)(x-4)$

Even multiplicity touches axis

Odd multiplicity crosses axis

$$f(x) = (x+5)^3 (x-3)^4$$

Find a) leading term x^7

b) degree 7

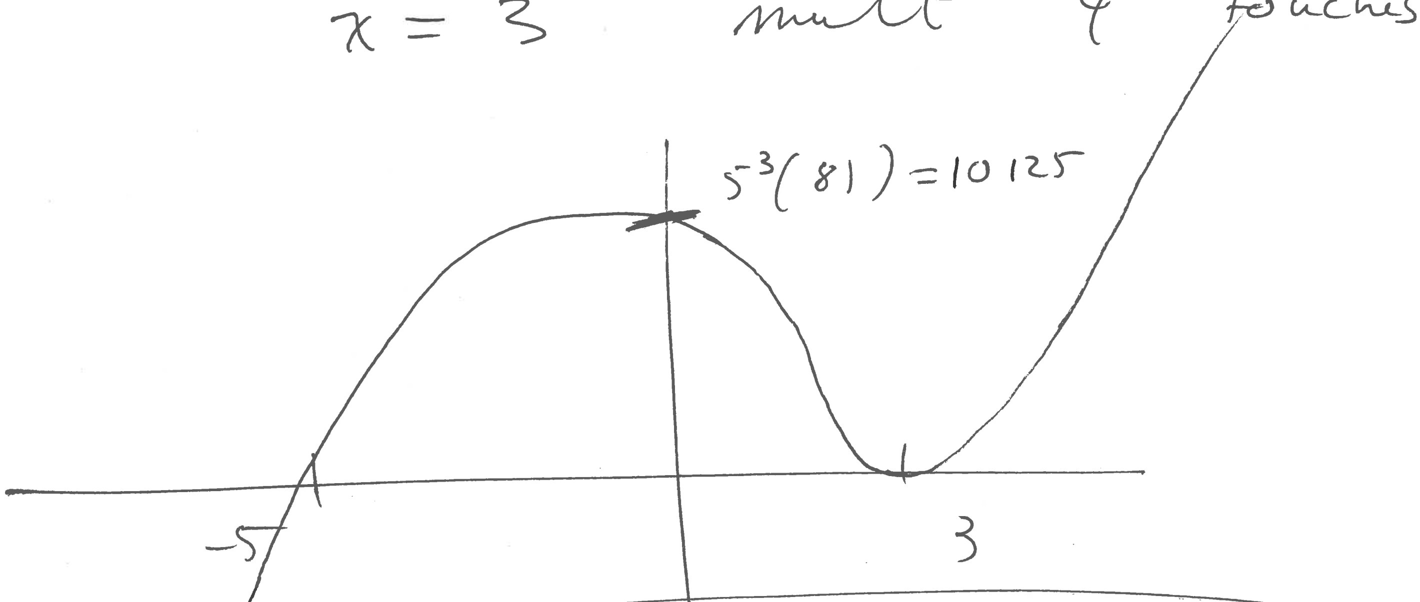
c) Zeros & multiplicity

d) Sketch graph

$$f(x) = (x+5)(x+5)(x+5)(x-3)(x-3)(x-3)(x-3)$$

zeros: $x = -5$ mult 3 crosses

$x = 3$ mult 4 touches



leading coefficient $= 1$

constant term $f(0) = 5^3 \cdot (-3)^4$

TEST Sketch

Ex: $f(x) = -2x^3(x+1)(x+2)^2$

Find: leading term $(-2x^3)(x)(x)^2 = \boxed{-2x^6}$

leading coefficient $\boxed{-2}$

degree $\boxed{6}$

constant term $\boxed{f(0)=0}$

zeros & mult

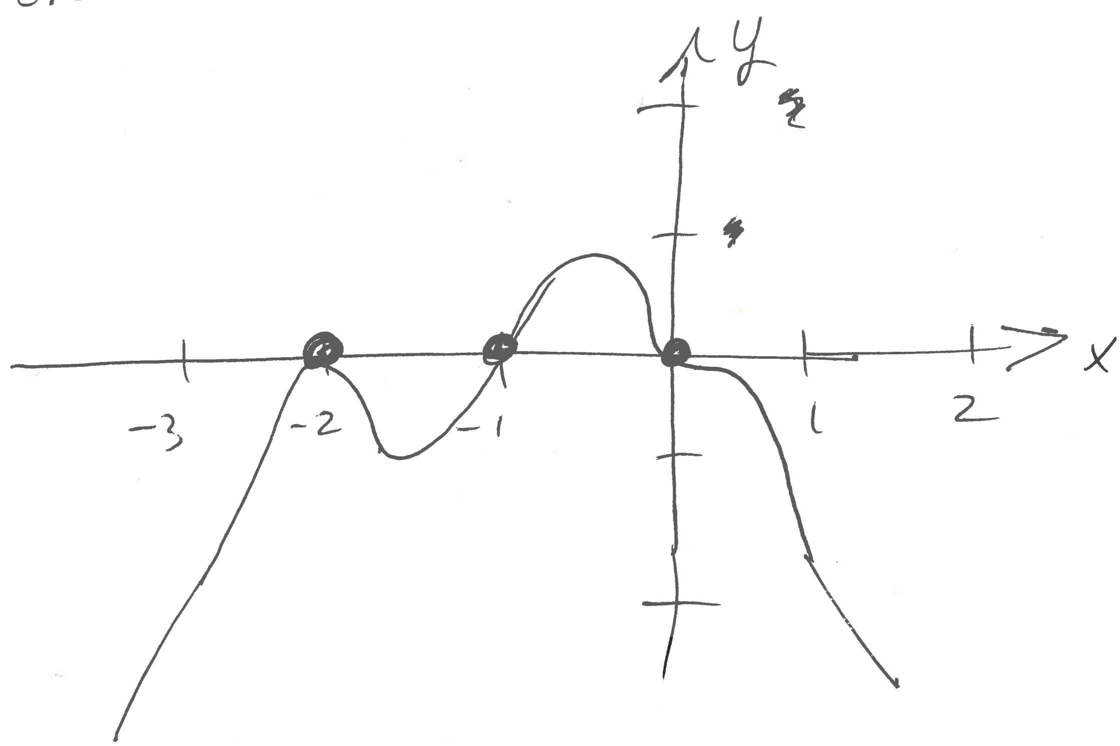
$-2x^3(x+1)(x+2)^2 = 0$

$x^3 = 0$ or $x+1=0$ or $(x+2)^2=0$

$x=0$ mult 3 crosses

$x=-1$ mult 1 crosses

$x=-2$ mult 2 touches.



Fundamental theorem of algebra

If $f(x)$ is a polynomial of degree n it has exactly n zeros up to multiplicity.

<u>Ex</u>: $f(x) = -2x^3(x+1)(x+2)^2$

degree 6 & has 6 zeros

$x = 0, 0, 0, -1, -2, -2$

3.2 Polynomial and Synthetic Division

We can use long division to divide polynomials the same way we use long division to divide integers.

Example 3.2.1. Simplify this ~~fraction~~: $\dfrac{x^4+5x^3+6x^2-x-2}{x+2}$. $= x^3+3x^2-1$

$$x^3+3x^2+0-1$$

$$x+2 \overline{\big)\, x^4+5x^3+6x^2-x-2}$$

$$-(x^4+2x^3)$$

$$3x^3+6x^2$$

$$-(3x^3+6x^2)$$

$$0-x-2$$

$$-(-x-2)$$

$$0$$

So we know that

$$(x+2)\left(\frac{x^4+5x^3+6x^2-x-2}{x+2}\right)=\left(x^3+3x^2-1.\right)(x+2)$$

More specifically we have a factor of $f(x)=x^4+5x^3+6x^2-x-2$.

$$\boxed{x^4+5x^3+6x^2-x-2=(x^3+3x^2-1)(x+2).}$$

Example 3.2.2. $\dfrac{5x^3+18x^2+8x-6}{x+3}$

$$
\begin{array}{r}
5x^2+3x-1 \\
x+3\,\overline{\big)\,5x^3+18x^2+8x-6} \\
-(5x^3+15x^2) \\
\hline
3x^2+8x \\
-(3x^2+9x) \\
\hline
-x-6 \\
-(-x-3) \\
\hline
-3
\end{array}
$$

remainder

So we know that

$$\frac{5x^3+18x^2+8x-6}{x+3} = 5x^2+3x-1+\frac{-3}{x+3}.$$

If we write it with no denominators we get an expression of the form:

$$\boxed{\underbrace{5x^3+18x^2+8x-6}_{P(x)} = \underbrace{(5x^2+3x-1)}_{q(x)}\underbrace{(x+3)}_{D(x)}\underbrace{-3}_{r(x)}}$$

This form illustrates the theorem known as **The Division Algorithm** which states that any two polynomials $P(x)$ and $D(x)$ (where $D(x)$ is of lower degree) can be written as

$$P(x) = D(x)\cdot q(x) + r(x)$$

More formally:

$$\frac{P(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}$$

The Division Algorithm
If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$ and the degree of $D(x)$ is less than or equal to the degree of $P(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$P(x) = D(x)\cdot q(x) + r(x)$$ ← No fractions

OR

$$\frac{P(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}$$

Synthetic Division (The easy way to divide)

$\boxed{x+a}$ must be dividing by this

Example 3.2.3. Divide using synthetic division: $\dfrac{5x^3+18x^2+8x-6}{x+3}$

add add add

-3	5	18	8	-6
		-15	-9	3
	5	3	-1	-3

two rows

← answer row

↖ remainder

$5x^2+3x-1 \leftarrow$ quotient

$$5x^3+18x^2+8x-6=(5x^2+3x-1)(x+3)-3$$

$P(-3)$

> **The Remainder Theorem**
> Suppose $p(x)$ is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x-c$ the remainder is $p(c)$.

Example 3.2.4. Use the remainder theorem to find the value of $p(-3)$ where

$$p(x)=5x^3+18x^2+8x-6$$

$$P(x)=5x^3+18x^2+8x-6=(5x^2+3x-1)(x+3)-3$$

$$P(-3)=(5x^2+3x-1)\underbrace{(-3+3)}_{0}-3$$

$$=\boxed{-3}$$

want to know $P(-3)$ then do synthetic division with -3.

-3 |

Example 3.2.5. Suppose we know that $x = -2$ is a zero of the polynomial $f(x) = 2x^3 + x^2 - 5x + 2$. Find all the zeros of the polynomial and write it in factored form.

Since we know that $x = -2$ is a zero then we know that

$$f(x) = 2x^3 + x^2 - 5x + 2 = (x+2) \cdot q(x)$$

(x + 2)

(x + 2)

and we can use synthetic division to find $q(x)$.

$$\begin{array}{r|rrrr} -2 & 2 & 1 & -5 & 2 \\ & \downarrow & -4 & 6 & -2 \\ \hline & \underbrace{2 \quad -3 \quad 1}_{q(x)} & & & 0 \end{array}$$

← remainder

$$2x^3 + x^2 - 5x + 2 = (x+2)(2x^2 - 3x + 1)$$

$$\boxed{= (x+2)(2x-1)(x-1)}$$

$$\boxed{\text{zeros} \quad x = -2, \quad x = \tfrac{1}{2}, \quad x = 1}$$

Example 3.2.6. Use synthetic division to divide $\frac{-3x^4}{x+2}$

$8x^2+2x-3=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-4(8)(-3)}}{2(8)}$$

$$=\frac{-2\pm 10}{16}=\frac{8}{16} \text{ or } \frac{-12}{16}$$

Example 3.2.7. Given that $(x+2)$ and $(x-4)$ are factors of $f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$ find all the zeros of $f(x)$.

$$\frac{8x^4-14x^3-71x^2-10x+24}{(x+2)(x-4)}=\frac{(x+2)(x-4)\,q(x)}{(x+2)(x-4)}$$

-2	8	-14	-71	-10	24	$\leftarrow x^4$
	↓	-16	60	22	-24	
+4	8	-30	-11	12	0	$\leftarrow 8x^3-30x^2-11x+12$
	↓	32	8	-12		
	8	2	-3	0		$\leftarrow 8x^2+2x-3$

$8x^2+2x-3=(4x+3)(2x-1)$

$8x^2-4x+6x-3$

$x=-2$ $\quad x=-\frac{3}{4}$ $\quad 4x+3=0$ $\quad 2x-1=0$

$x=4$ $\quad x=\frac{1}{2}$

$x=-\frac{3}{4}$

$4x=-3$

$4x+3=0$

Example 3.2.8. Create a polynomial p which has the desired characteristics. Leave the polynomial in factored form.

- degree 4
- root of multiplicity 2 at $x = 1$
- roots of multiplicity 1 at $x = 0$ and $x = -3$.
- the graph passes through the point (2,20)

$$A(x-1)(x-1)(x+3)x = p(x)$$

$$p(2) = 20 = A(2-1)(2-1)(2+3)(2)$$

$$\frac{20}{10} = \frac{10A}{10}$$

$$2 = A$$

$$\boxed{p(x) = 2x(x-1)^2(x+3)}$$

3.3 Real Zeros of a Polynomial

The Rational Roots Test
If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and
p = a factor of the constant term a_0
q = a factor of the constant term a_n.

Example 3.3.1. Use the rational roots test to find all the real zeros of

1. $g(x) = x^3 - 4x^2 - x + 4$

$\dfrac{\text{Factors of } 4}{\text{Factors of } 1} = \dfrac{\pm 1, \pm 2, \pm 4}{\pm 1}$

$$\begin{array}{c|cccc} 1 & 1 & -4 & -1 & 4 \\ & & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$ ✓

$x^3 - 4x^2 - x + 4 = (x-1)(x^2 - 3x - 4)$ ✓

$= (x-1)(x-4)(x+1)$

2. $p(x) = x^3 + 7x^2 + 9x - 5$

$\dfrac{\text{Factors of } 5}{\text{Factors of } 1} = \dfrac{\pm 1, \pm 5}{\pm 1}$

try
$$\begin{array}{c|cccc} 1 & 1 & 7 & 9 & -5 \\ & & 1 & 8 & 17 \\ \hline & 1 & 8 & 17 & \end{array}$$ Not zero

try
$$\begin{array}{c|cccc} -5 & 1 & 7 & 9 & -5 \\ & & -5 & -10 & 5 \\ \hline & 1 & 2 & -1 & 0 \end{array}$$ ✓

$x^2 + 2x - 1 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$

$= \dfrac{-2 \pm \sqrt{8}}{2}$

$\dfrac{-2 \pm 2\sqrt{2}}{2} = \boxed{-1 \pm \sqrt{2}}$

HW

$$\frac{2x^3-16x^2+7x-38}{2x^2+5} = f(x)$$

$$\begin{array}{r l}
 & x - 8 \\
2x^2+5 \;\big) & 2x^3 - 16x^2 + 7x - 38 \\
 & -(2x^3 + 0x^2 + 5x) \leftarrow x(2x^2+5) \\
\hline
 & -16x^2 + 2x - 38 \\
 & -(-16x^2 + 0x - 40) \leftarrow -8(2x^2+5) \\
\hline
 & 2x + 2
\end{array}$$

$$\boxed{x - 8 + \frac{2x+2}{2x^2+5} = f(x)}$$

HW

$$f(x) = \frac{x^3 - 6x^2 + 13}{x-2}$$

0x

Divide: synthetic division

$$\begin{array}{r|rrrr} +2 & 1 & -6 & 0 & 13 \\ & \downarrow & 2 & -8 & -16 \\ \hline & 1 & -4 & -8 & -3 \end{array}$$

remainder

$$\boxed{f(x) = x^2 - 4x - 8 + \frac{-3}{x-2}}$$

$$\underbrace{x^3 - 6x^2 + 13}_{} = \underbrace{(x^2-4x-8)}_{q(x)}\underbrace{(x-2)}_{d(x)} \underbrace{- 3}_{+\, r}$$

Long division

$$\begin{array}{r|l} & x^2 - 4x - 8 \quad q(x) \\ \hline x-2 & x^3 - 6x^2 \quad + 13 \\ & -(x^3 - 2x^2) \downarrow \\ \hline & -4x^2 + 13 \\ & -(-4x^2 + 8x) \\ \hline & -8x + 13 \\ & -(-8x + 16) \\ \hline & -3 \end{array}$$

$x^2(x-2)$

$-4x(x-2)$

$-8(x-2)$

$r = -3$

3.4 Complex Numbers

Q: What is a complex number?

A: It is a number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

- a is called the **real part**
- b is called the **imaginary part**.
- The **conjugate** of $a + bi$ is $a - bi$.

$1+\sqrt{2}$ conjugate is $1-\sqrt{2}$

Properties

Ex: $7 = 7 + 0i$

$1+2i$, $3i$, i

$\frac{1}{2}+\sqrt{2}\,i$

$\sqrt{3}-14i$

$14i-\sqrt{3}$

1. $a + b\,i = c + d\,i \quad \Leftrightarrow \quad a = c$ and $b = d$
2. $(a + b\,i) + (c + d\,i) = (a + c) + (b + d)\,i$
3. $(a + b\,i)\cdot(c + d\,i) = ac + ad\,i + bc\,i + bd\,i^2 = (ac - bd) + (ad + bc)\,i$

   $i^2 = -1$
4. $(a + b\,i)(a - b\,i) = a^2 + b^2$

$(a+bi)(a-bi) = a^2 - abi + abi - b^2i^2$

$= a^2 - b^2 i^2$

$= a^2 - b^2(-1)$

$= a^2 + b^2$

Example 3.4.1.

a. $(4 + i) + (5 + 3i) = 9 + 4i$

b. $(2i + 7) - 2i = 7$

c. $(3 + 2i) + (4 - i) - (7 + i) = 0$

d. $(5 + 2i)(4 - 3i) = 20 - 15i + 8i - 6i^2$

$= 20 - 7i + 6$

$\boxed{= 26 - 7i}$

$i^2 = -1$

e. $\dfrac{i}{3 + i}$

$$\frac{(i)}{(3+i)}\cdot\frac{(3-i)}{(3-i)} = \frac{3i - i^2}{9 - 3i + 3i - i^2} = \boxed{\frac{3i+1}{10}}$$

$-i^2 = -(-1) = +1$

write in $a+bi$ form.

$\boxed{\frac{1}{10} + \frac{3}{10}i}$

$$\begin{cases} i^0 \\ i^2 = -1 \\ i^3 = i^2 \cdot i = -i \\ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \end{cases}$$

$\sqrt{-1}$

$$\begin{cases} i^5 = i^4 \cdot i = i \\ i^6 = i^4 \cdot i^2 = -1 \\ i^7 = \boxed{-i} \\ i^8 = \boxed{1} \end{cases}$$

$i^{n4} = \boxed{1}$

$i^{25} = i^{24} \cdot i = \boxed{i}$

$i^{1003} = i^{1000} \cdot i^3 = \boxed{-i}$

$i^2 = -1$

f. $\dfrac{(3-5i)}{(2-i)} \dfrac{(2+i)}{(2+i)} = \dfrac{6+3i-10i-5i^2}{4+1} = \boxed{\dfrac{11-7i}{5}}$

g. $(3-\sqrt{-4})+(-8+\sqrt{-25}) = 3 - i\sqrt{4} + (-8 + i\sqrt{25})$
$= 3 - 2i - 8 + 5i = \boxed{-5+3i}$

h. $(\sqrt{-5})(\sqrt{-5}) = (i\sqrt{5})(i\sqrt{5}) = i^2(\sqrt{5})^2 = \boxed{-5}$

i. $(2-\sqrt{-1})(5+\sqrt{-9}) = (2-i)(5+3i)$
$= 10 + 6i - 5i - 3i^2$ $i^2=-1$
$\boxed{= 13 + i}$

j. $\cancel{\dfrac{1}{3i}}$

$\dfrac{1}{3i} \cdot \dfrac{i}{i} = \dfrac{i}{3i^2} = \boxed{-\dfrac{i}{3}}$

Complex numbers in quadratic equations

Example 3.4.2. Solve for x: $x^2 + 6x + 10 = 0$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \dfrac{-6 \pm \sqrt{36-4(1)(10)}}{2(1)}$

$= \dfrac{-6 \pm \sqrt{-4}}{2} = \dfrac{-6 \pm 2i}{2} = \boxed{-3 \pm i}$

$x^2+6x+10 = (x-(-3+i))(x-(-3-i))$

Example 3.4.3. Solve for x: $x^2 + 1 = 0$

$x^2+1 = (x+i)(x-i)$

$-1 \quad -1$

$x^2 = -1$

$x = \pm\sqrt{-1}$

$\boxed{x = \pm i}$

Example 3.4.4. Solve for x: $9x^2 - 6x + 37 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{6 \pm \sqrt{36 - 4(9)(37)}}{2(9)} = \frac{6 \pm \sqrt{36 - 36(37)}}{18} = \frac{6 \pm \sqrt{36(1-37)}}{18} = \frac{6 \pm \sqrt{-1296}}{18} = \frac{6 \pm i\sqrt{1296}}{18}$$

$$= \frac{6 \pm 36i}{18} \qquad \boxed{= \frac{6 \pm 36i}{18}}$$

The Linear Factorization Theorem
If $f(x)$ is a polynomials of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Example 3.4.5. Find all the zeros of

1. $f(x) = (x + 5)(x - 8)^2$

   $x = -5, 8, 8$

2. $f(t) = (t - 3)(t - 2)(t - 3i)(t + 3i) = 0$

   $t - 3 = 0 \quad t - 2 = 0 \quad t - 3i = 0 \quad t + 3i = 0$

   $\boxed{t = 3 \quad t = 2 \quad t = 3i \quad t = -3i}$

Ex 3. $f(x) = (x - 3 + 2i)(x - 3 - 2i)$

$x - 3 + 2i = 0$ or $x - 3 - 2i = 0$

$\boxed{x = 3 - 2i \qquad x = 3 + 2i}$

Example 3.4.6. Find a polynomial function of degree 7 with integer coefficients that has ONLY the zeros $c = 4$, $c = -3i$ and $c = 3i$ Leave factored.

$$\boxed{f(x) = (x-4)^3 (x+3i)^2 (x-3i)^2}$$

Example 3.4.7. Find all zeros of $f(x) = 4x^3 + 12x^2 + 11x + 6$ given that $x = -2$ is one of the zeros.

$$\frac{4x^3 + 12x^2 + 11x + 6}{x+2} = \frac{(x+2)\, g(x)}{x+2}$$

-2	4	12	11	6
	$\downarrow$	-8	-8	-6
	4	4	3	0

$$\sqrt{32} = \sqrt{16} \cdot \sqrt{2}$$

$$4x^2 + 4x + 3 = g(x)$$

$a = 4$
$b = 4$
$c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{16 - 4(4)(3)}}{2(4)} = \frac{-4 \pm \sqrt{-32}}{8} = \boxed{\frac{-4 \pm i\sqrt{32}}{8}}$$

$$= \frac{-4 \pm i\, 4 \cdot \sqrt{2}}{8} = \boxed{\frac{-1 \pm i\sqrt{2}}{2}}$$

Example 3.4.8. Find all the roots of $f(x) = x^3 - 7x^2 - x + 87$ knowing that $5 + 2i$ is a zero.

$\underbrace{5+2i \text{ is a zero}}$ so $5-2i$ is a zero.

$x = 5+2i$ ← all real parts on left

$(x-5)^2 = (2i)^2$ ← square both sides

$x^2 - 10x + 25 = -4$

$\underline{x^2 - 10x + 29 = 0}$

don't know

$x^3 - 7x^2 - x + 87 = (x^2 - 10x + 29)(x + A)$

$= x^3 + \underbrace{Ax^2 - 10x^2} - 10Ax + 29x + 29A$

$Ax^2 - 10x^2 = -7x^2$

$A - 10 = -7$

$A = 3$

$f(x) = (x^2 - 10x + 29)(x+3)$

3 zeros $\boxed{\begin{aligned} x &= 5+2i \\ x &= 5-2i \\ x &= -3 \end{aligned}}$