2 Linear and Quadratic Functions

2.1 Linear Equations in Two Variables

The simplest mathematical model is the **linear equation in two variables.** The standard form is (**slope-intercept**)

$$\boxed{y = mx + b}$$

where m is the slope and b is the y-intercept. You will recall that

$$\text{slope} = \frac{\text{rise}}{\text{run}}.$$

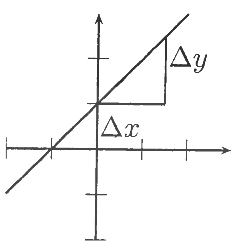
Δ means "change in"

The slope is the amount of vertical change relative to the horizontal change. Sometimes we think of it as the "change in y" over "change in x".

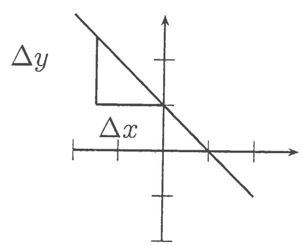
To calculate the slope between two points (x_1, y_1) and (x_2, y_2) the formula is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}.$$

Positive Slope



Negative Slope

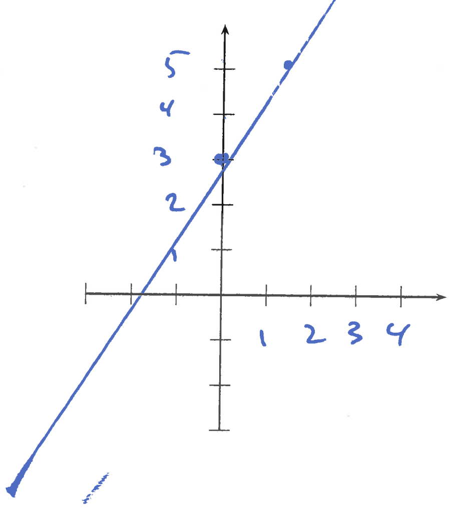


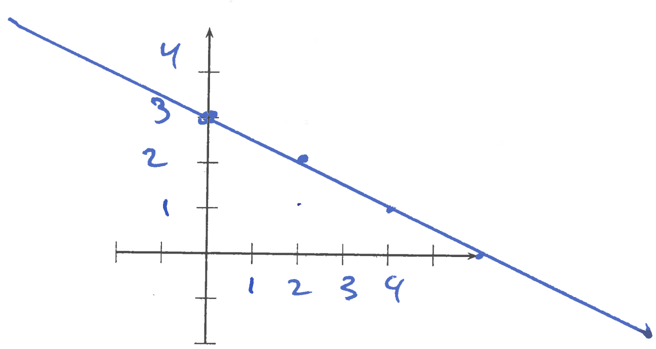
$\text{slope} = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$

$\text{Slope} = \frac{-1}{2} = \frac{\text{rise}}{\text{run}}$

Example 2.1.1. Sketch the graphs of the following two functions:

$$y = 2x + 3 \qquad y = -\tfrac{1}{2}x + 3. \qquad = \frac{1}{-2}$$





Example 2.1.2. Find the slope between the following pairs of points.

(x_1, y_1) (x_2, y_2)

(a). (-3, 0) and (4, 4)

(b). (-3, 1) and (4, 1)

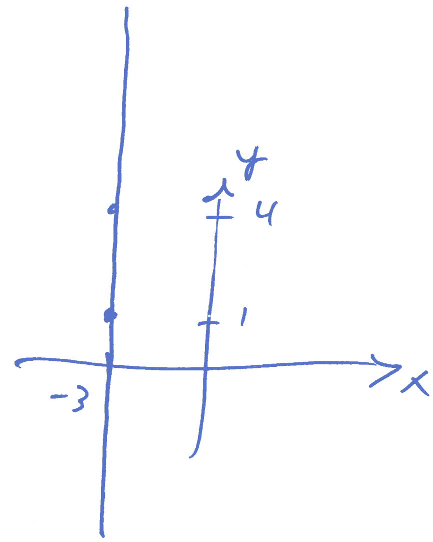
(c). (-3, 1) and (-3,4)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4-0}{4-(-3)} = \frac{0-4}{-3-4} = \frac{+4}{7}$$

$$m = \frac{1-1}{4-(-3)} = \frac{1-1}{-3-4} = 0$$

$$m = \frac{4-1}{-3-(-3)} = \frac{3}{0} \text{ undefined}$$



2.1.1 Point-Slope Form

You always need **two** things:

1. a point: (x_1, y_1) AND
2. a slope m.

$$y - y_1 = m(x - x_1)$$

Variables.
Do NOT put numbers here.

TEST **Example 2.1.3.** Write the equation of the line through (-3,0) and (4, -4). Write the equation in the point slope form and the slope-intecept form.

(x_1, y_1) (x_2, y_2)

Need 1. pt (-3, 0)

2. slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-0}{4-(-3)} = -\frac{4}{7}$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 0 = -\frac{4}{7}(x - (-3))}$$ point slope.

$$\boxed{y = -\frac{4}{7}x - \frac{12}{7}}$$ slope intercept

Example 2.1.4. Write the equation of the lines through

(a). (-3, 1) and (4, 1) $m = 0$

(b). (-3, 1) and (-3,4) $m =$ undef.

(a) $y - y_1 = m(x - x_1)$

$y - 1 = 0(x - 4)$

$y - 1 = 0$

$\boxed{y = 1}$ ← Horizontal line

$y =$ some number.

(b) $\boxed{x = -3}$ Vertical line

$x =$ some number

2.1.2 Parallel and Perpendicular Lines

Parallel lines have the same slope. If $y = m_1\, x + b_1$ is parallel to $y = m_2\, x + b_2$ then $m_1 = m_2$.

Perpendicular lines have negative reciprocal slopes. If $y = m_1\, x + b_1$ is perpendicular to $y = m_2\, x + b_2$ then $m_1 = -\dfrac{1}{m_2}$.

Example 2.1.5. Write the equations of the lines parallel and perpendicular to $-4x + 2y = 3$ passing through the point (2, 1).

Need 1. pt $(2,1) = (x_1, y_1)$ ✓

2. Slope $m =$ don't know yet. ✗

Write $-4x + 2y = 3$ in $y = mx + b$ form.

$$\begin{aligned} -4x + 2y &= 3 \\ +4x \qquad &\quad +4x \\ \hline \frac{2y}{2} &= \frac{4x}{2} + \frac{3}{2} \end{aligned} \Longrightarrow y = 2x + \frac{3}{2}$$

original line

$m_{\parallel} = 2$ $\quad y - y_1 = m(x - x_1)$

parallel line:

$\boxed{y - 1 = 2(x - 2)}$ ← point slope

$\boxed{y = 2x - 3}$ ← slope intercept

$m_{\perp} = -\frac{1}{2}$ $\quad y - y_1 = m(x - x_1)$

want $y = mx + b$

$y = -\frac{1}{2}x + b$ ← put in (2,1) here to solve for b.

$1 = -\frac{1}{2}(2) + b$

$1 = -1 + b \Longrightarrow b = 2$

$\boxed{y = -\frac{1}{2}x + 2}$

2.2 Absolute Value Functions

Definition 2.1. The absolute value of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Absolute Value Properties

1. **Product rule:** $|ab| = |a||b|$
2. **Power rule:** $|a^n| = |a|^n$
3. **Quotient rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. **Equality property 1:** $|x| = 0$ if and only if $x = 0$
5. **Equality property 2:** For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
6. **Equality property 1:** For $c < 0$, $|x| = c$ has no solution.

An equation with an absolute value is always TWO equations:

$$|\, x \,| = 4 \qquad \Longrightarrow \qquad x = 4 \text{ OR } -x = 4$$

$x = -4$

Example 2.2.1. $|\, x \,| = x^2 + x - 3$

We start by writing it as two equations:

$$x = x^2 + x - 3$$
$$-x \qquad -x$$
$$0 = x^2 - 3$$
$$+3 \qquad +3$$
$$3 = x^2$$
$$\pm\sqrt{3} = x$$

$$-x = x^2 + x - 3$$
$$+x \qquad +x$$
$$0 = x^2 + 2x - 3$$
$$0 = (x+3)(x-1)$$
$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -3 \qquad \cancel{x = 1}$$

ALWAYS check your solutions:

$\boxed{x = +\sqrt{3}}$ $\quad |\sqrt{3}| = (\sqrt{3})^2 + \sqrt{3} - 3 = \sqrt{3}$ ✓

$x = -\sqrt{3}$ $\quad |-\sqrt{3}| = (-\sqrt{3})^2 - \sqrt{3} - 3 = -\sqrt{3}$ NO

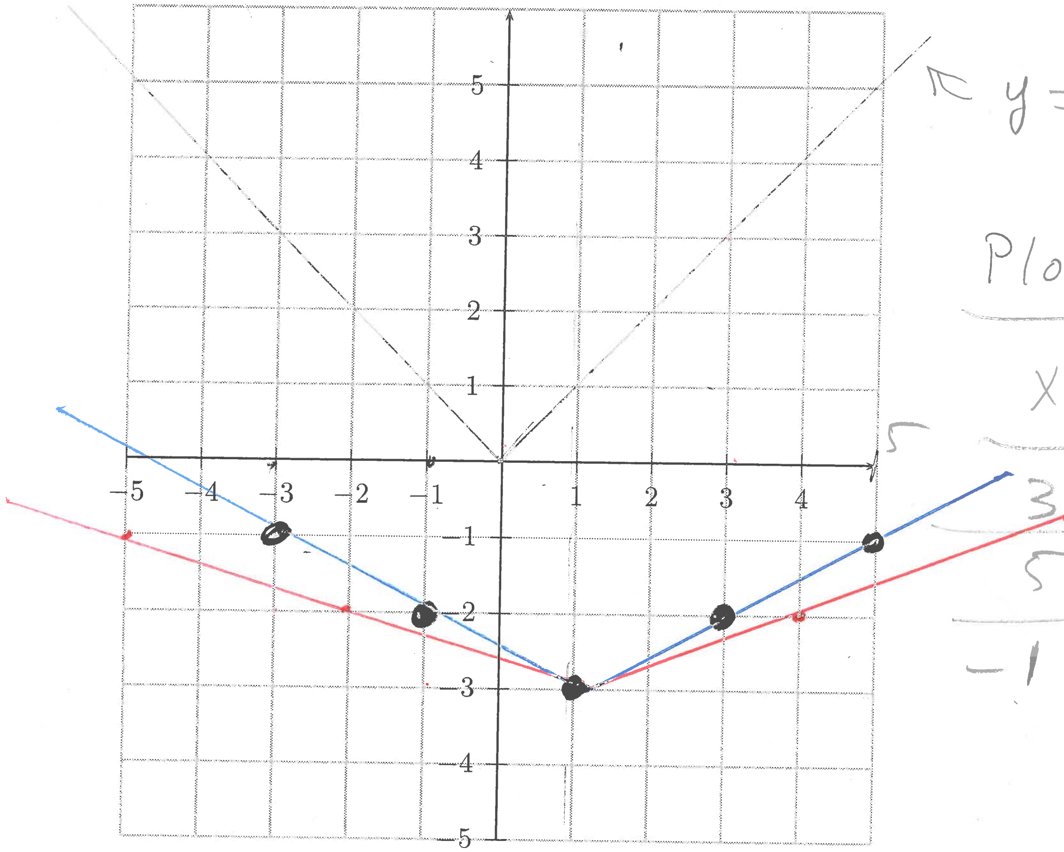
$x = 1$ $\quad |1| = 1^2 + 1 - 3 = -1$ NO.

$\boxed{x = -3}$ $\quad |-3| = 9 - 3 - 3 = 3$ ✓

Example 2.2.2. Sketch a graph of $f(x) = \frac{1}{2}|x - 1| - 3$

right 1

down 3



$y = |x|$

Plot points

x	$y = \frac{1}{2}\lvert x-1\rvert - 3$
3	$y = \frac{1}{2}\lvert 2\rvert - 3 = -2$
5	$y = \frac{1}{2}\lvert 4\rvert - 3 = -1$
−1	$y = \frac{1}{2}\lvert -1-1\rvert - 3 = \frac{1}{2}\lvert -2\rvert - 3 = -2$
−3	$y = \frac{1}{2}\lvert -3-1\rvert - 3 = \frac{1}{2}\lvert -4\rvert - 3 = \frac{1}{2}(4) - 3 = -1$

$y = \frac{1}{3}|x-1| - 3$

2.3 Quadratic Functions

Definition 2.2. Let a, b, and c be real numbers with $a \neq 0$. The function

$$f(x) = ax^2 + bx + c = y$$

is called a **quadratic equation**.

The graph of a quadratic equation is a parabola. All parabolas are symetric with respect to the axis of symetry which passes through the vertex.

Example 2.3.1.

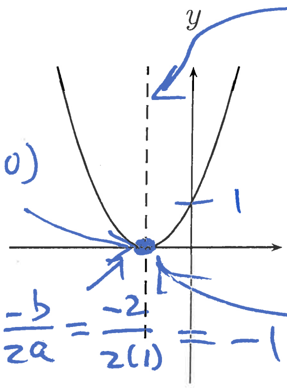
$f(x) = x^2 + 2x + 1$ OR $f(x) = -x^2 - 2x - 1$

$\boxed{x = \text{some number}}$

axis of symmetry

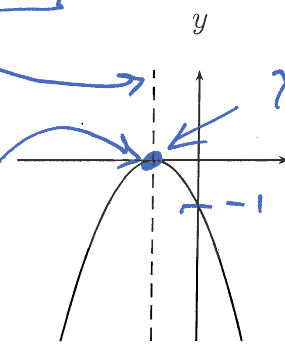
vertex

(−1, 0)



$x = \frac{-b}{2a} = \frac{-2}{2(1)} = -1$

$a > 0$ graph opens up



$x = \frac{-b}{2a} = \frac{+2}{2(-1)} = -1$

y-intercept? Set $x = 0$

$a < 0$ graph opens down

The Standard Form of a Parabola

$$f(x) = a(x - h)^2 + k = y$$

Vertex is located at (h, k)

if $a > 0$ graph opens up
if $a < 0$ graph opens down

Alternate form

If

$$f(x) = ax^2 + bx + c$$

then the

$$\text{vertex} = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right).$$

x-coord of vertex $= \frac{-b}{2a}$

graph $f(x) = 2x^2 - 12x - 14$

$a = 2$
$b = -12$
$c = -14$

Step 1: vertex $x = \frac{-b}{2a}$

$= \frac{-(-12)}{2(2)} = \frac{12}{4} = 3$

$y = f(3) = 2(9) - 12(3) - 14$
$= -32$

$(3, -32)$ Vertex

Step 2: x-int set $y = 0$

$0 = 2x^2 - 12x - 14$
$0 = 2(x^2 - 6x - 7)$
$0 = 2(x - 7)(x + 1)$
$2(x^2 - 7x + x - 7)$
$0 = 2(x+1)(x-7)$

$x + 1 = 0$ or $x - 7 = 0$
$x = -1$ $x = 7$

y-int $f(0) = -14$

Example 2.3.3. Graph $f(x) = (x-6)^2 + 3$ $= (x-h)^2 + k$

Vertex $(6,3)$

x-int: $0 = (x-6)^2 + 3$

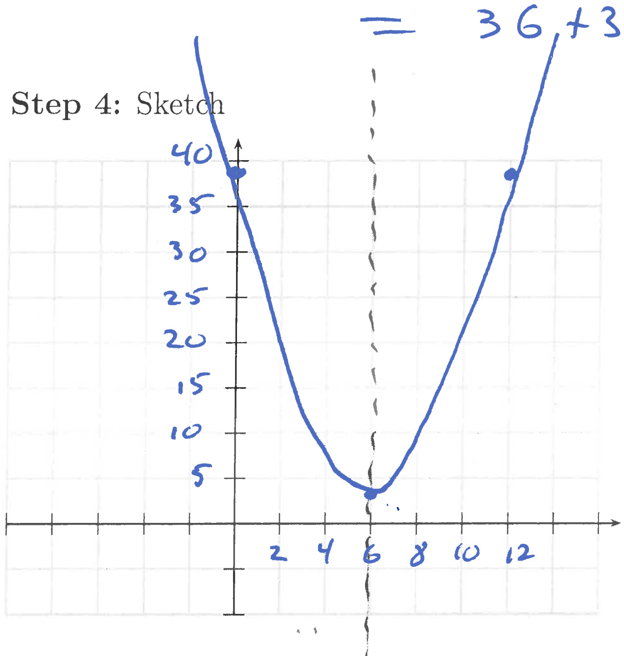
$-3 \qquad -3$

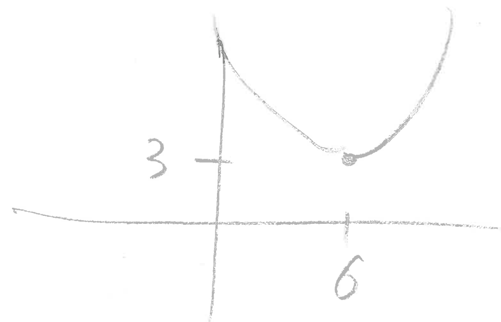
$\pm\sqrt{-3} = \sqrt{(x-6)^2}$ No answer. No x-int.

y-int $f(0) = (0-6)^2 + 3$

$= 36 + 3 = 39$

Step 4: Sketch





Example 2.3.4. Find the standard form for a parabola that has (0,1) as its vertex and passes through the point (1,0)

$y = f(x) = a(x-h)^2 + k$ $(h,k) = \text{vertex}$

$(h,k) = (0,1)$

$y = f(x) = a(x-0)^2 + 1$

Stick the point $(1,0)$ into the equation.

$0 = a(1)^2 + 1$

$a = -1$

$y = (-1)(x-0)^2 + 1$

$\boxed{y = -x^2 + 1}$

Complete the square

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$(x-a)^2 = x^2 - 2ax + a^2$$

leave space

$$x^2 + 4x + 7 = x^2 + 4x + \underline{(2)^2} + 7 \underline{\ -4\ }$$

half of the number in front of x, squared.

$$= (x+2)^2 + 3$$

$$x^2 + 4x + 4 = (x+2)(x+2)$$

Example 2.3.5. Convert to standard form $f(x) = x^2 + 6x + 5$ and graph.

$f(x) = a\underbrace{(x-h)^2} + k$

Vertex $(-3, -4)$

opens up

complete the square

x-int: set $y = 0$

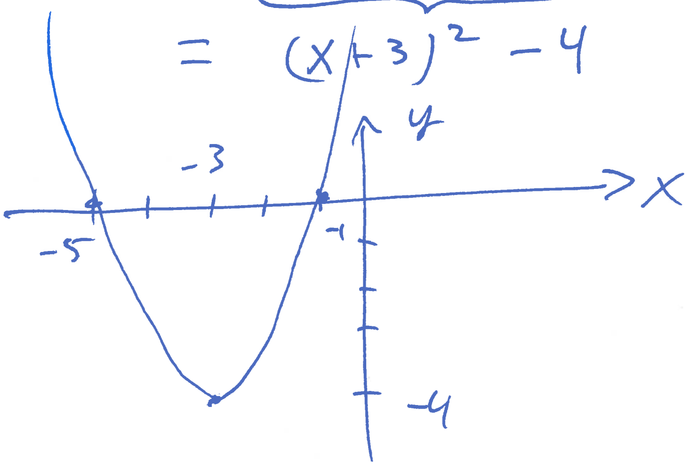
$f(x) = \underbrace{x^2 + 6x + (3)^2} + 5 - 9$

$= (x+3)^2 - 4$

$x^2 + 6x + 5 = 0$

$(x+5)(x+1) = 0$

$x = -5, x = -1$



Example 2.3.6. Convert to standard form $f(x) = x^2 - 2x - 8$ and graph.

complete the square

$f(x) = (x-h)^2 + k$

$f(x) = x^2 - 2x + 1 - 8 - 1$

$\boxed{f(x) = (x-1)^2 - 9}$ ← vertex $(1, -9)$

x-int: $0 = (x-1)^2 - 9$

$+9 \qquad +9$

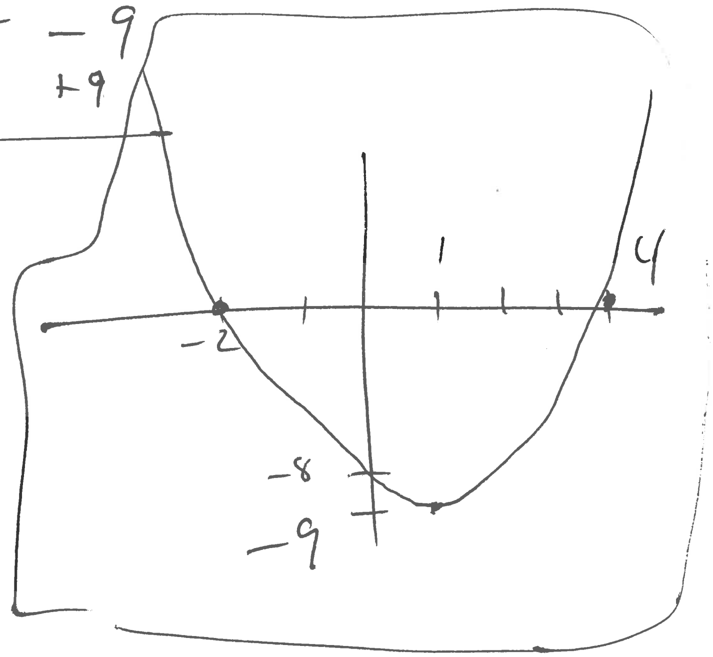
$\pm\sqrt{9} = \sqrt{(x-1)^2}$

$\pm 3 = x - 1$

$+1 \qquad +1$

$1 \pm 3 = x$

$x = 4, -2$



$y = 4x^2 - 10x - 11$

$a = 4$

$b = -10$

$c = -11$

y-int (vertical)

x-int

Vertex $x = \frac{-b}{2a} = \frac{10}{2(4)} = \frac{5}{4}$

$\boxed{\left(\frac{5}{4}, -18.5\right)}$ $y = -18.5$

y-int set $x=0$: $y = -11$

$\boxed{(0, -11)}$

x-int set $y=0$

quadric formula: $0 = ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(4)(-11)}}{2(4)}$$

$$= \frac{10 \pm \sqrt{100 + 176}}{8} = \frac{10 \pm \sqrt{276}}{8}$$

$$= \frac{10 \pm \sqrt{4 \cdot 69}}{8} = \frac{10 \pm 2\sqrt{69}}{8} = \frac{10}{8} \pm \frac{2\sqrt{69}}{8}$$

$$= \frac{5}{4} \pm \frac{\sqrt{69}}{4}$$

16 × 16 = 176

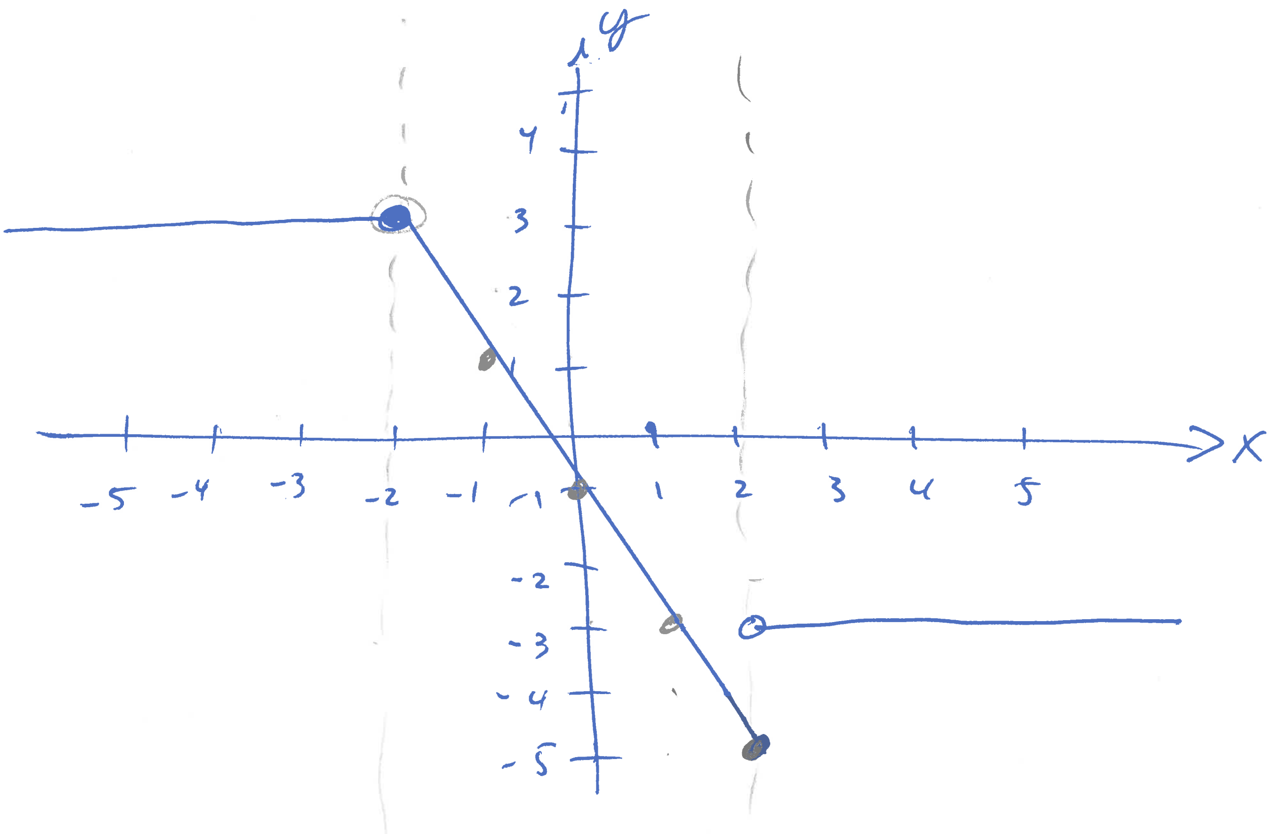
$4 \overline{)276}$ = 69

$y=3$ Horizontal line

§1.6

#15

$$f(x) = \begin{cases} 3 & -2 \geq x \\ -2x-1 & -2 < x \leq 2 \\ -3 & x > 2 \end{cases}$$



Example 2.3.7. Factor the following:

1. $9x \cdot (x-3) + 7 \cdot (x-3)$

$$\boxed{= (x-3)(9x+7)}$$

$15 = 5 \cdot 3$
$= 15 \cdot 1$

2. $x^3 - 8x^2 + 15x = x(x^2 - 8x + 15)$

$$\boxed{= x(x-5)(x-3)}$$

3. $x^2 + 11x + 28 \boxed{= (x+7)(x+4)}$

$6 = 6 \cdot 1$
$= 3 \cdot 2$

$4 = 2 \cdot 2$
$= 4 \cdot 1$

4. $6t^2 + 5t - 4 \boxed{= (3t+4)(2t-1)}$

5. $5z^2 + 23z + 12 \boxed{= (5z+3)(z+4)}$

$12 = 12 \cdot 1$
$= 2 \cdot 6$
$= 4 \cdot 3.$

$5 = 5 \cdot 1$

2.4 Solving Inequalities with Absolute Value and Quadratic Functions

2.4.1 Graphing inequalities

$x > 2$

$x \geq 2$

$x \geq 2$ AND $x \leq 4$

overlap

$x \geq 2$

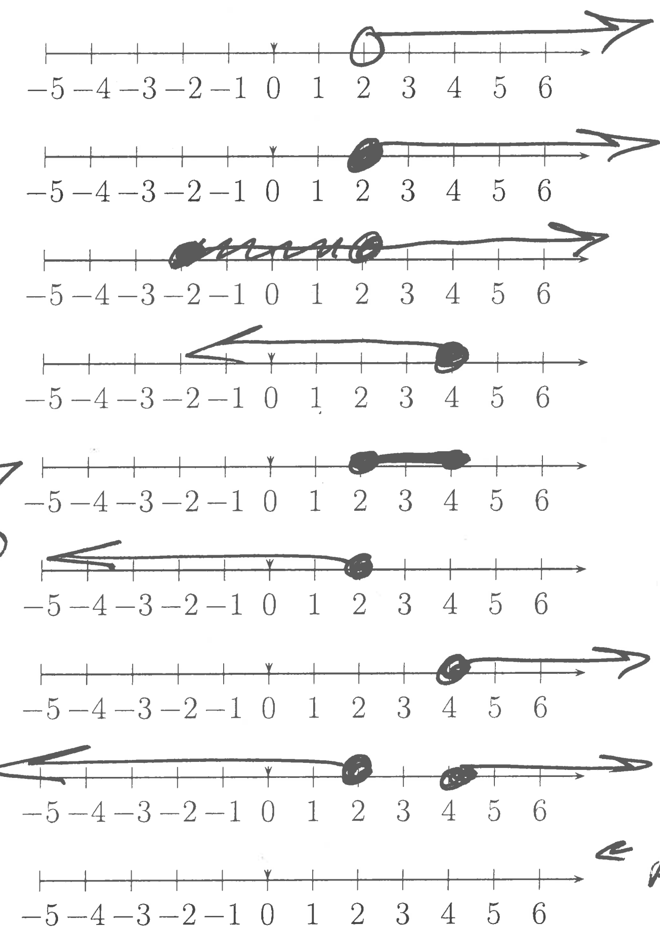
$x \leq 4$

$x \leq 2$ OR $x \geq 4$

both regions

$x \leq 2$

$x \geq 4$

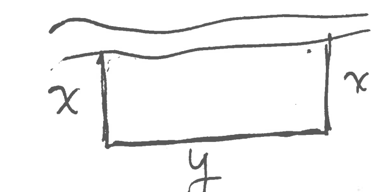


$x \leq 2$ AND $x \geq 4$ ← Answer

2.4.2 Interval Notation

Inequality notation		Interval notation
$x > 2$		$(2, \infty)$
$x \geq 2$		$[2, \infty)$
$x \geq 2$ AND $x \leq 4$	$2 \leq x \leq 4$	$[2, 4]$
$x \leq 2$ OR $x \geq 4$		$(-\infty, 2] \cup [4, \infty)$

Example 2.3.8. A rancher has 260 yards of fence with which to enclose three sides of a rectangular meadow (the fourth side is a river and will not require fencing). Find the dimensions of the meadow with the largest possible area.

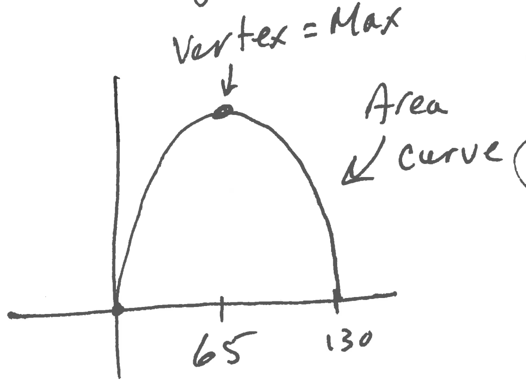


$2x + y = 260 \leftarrow$ solve for $y = \underline{260 - 2x}$

$A = xy$ $\quad x = \frac{-b}{2a}$ Vertex

$A = x(260 - 2x) = 260x - 2x^2$

Vertex $x = \frac{-260}{-2(2)}$
$(a)\ 2$



$x = 65$

$y = 260 - 2(65)$

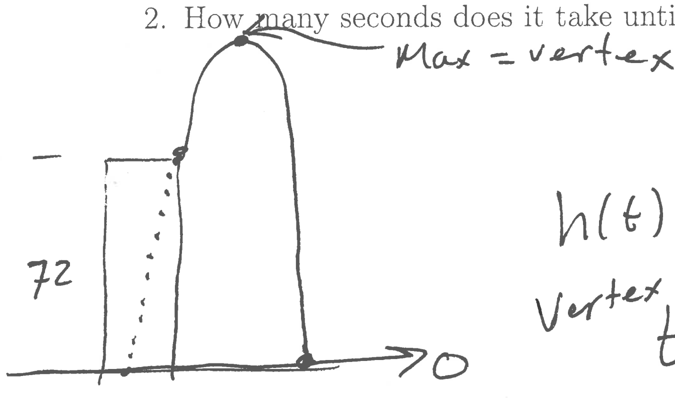
$y = 130$

$\boxed{65 \times 130}$

Example 2.3.9. A person standing close to the edge on top of a 72-foot building throws a ball vertically upward. The quadratic function $h(t) = -16t^2 + 84t + 72$ models the ball's height about the ground, $h(t)$, in feet, t seconds after it was thrown.

1. What is the maximum height of the ball?

2. How many seconds does it take until the ball hits the ground?



$-16t^2 + 84t + 72$

$h(t) = -4(4t^2 - 21t - 18)$

Vertex $t = \frac{-b}{2a} = \frac{21}{2(4)} = \frac{21}{8}$

#1✓

250.5 = Some number. $= h\left(\frac{21}{8}\right) = -16\left(\frac{21}{8}\right)^2 + 84\left(\frac{21}{8}\right) + 72$

#2) $h(t) = 0$ = ground $= -4(4t^2 - 21t - 18)$

$0 = -4(4t + 3)(t - 6)$

$4t + 3 = 0$ or $t - 6 = 0$

$t = \frac{-3}{4}$ $\quad \boxed{t = 6}$

Properties of Inequalities

1. Transitive: $a < b$ and $b < c \implies a < c$
2. Addition of Constants: If $a < b$ then $a + c < b + c$
3. Addition: If $a < b$ and $c < d$ then $a + c < b + d$
4. Multiplication by a constant:

   If $c > 0$ and $a < b$ then $ac < bc$

   If $c < 0$ and $a < b$ then $ac > bc$ reverse inequality when multiply by negative.

NOTE: If you multiply or divide by a negative number you reverse the order of the inequality.

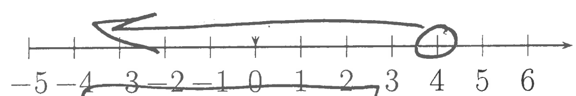
2.4.3 Solving linear inequalities

x raised to 1st power.

Example 2.4.1. $10x < 40$

$$\frac{10x}{10} < \frac{40}{10}$$

$$x < 4$$

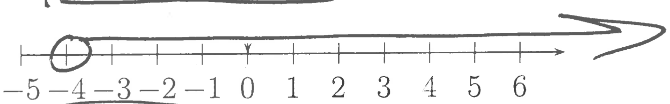


$\boxed{(-\infty, +4)}$

Example 2.4.2. $-10x < 40$

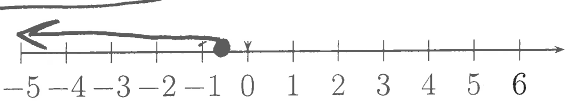
$$\frac{-10x}{-10} > \frac{40}{-10}$$

$\boxed{x > -4}$



$\boxed{(-4, \infty)}$

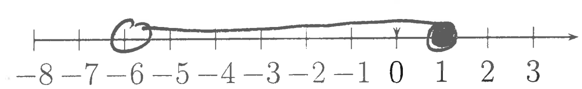
Example 2.4.3. $4(x + 1) \leq 2x + 3$



$$4x + 4 \leq 2x + 3$$
$$-2x - 4 \quad -2x - 4$$
$$2x \leq -1$$
$$\boxed{x \leq -\tfrac{1}{2}}$$

$(-\infty, -\tfrac{1}{2}]$

Example 2.4.4. $(-8 \leq -(3x + 5) < 13)(-1)$

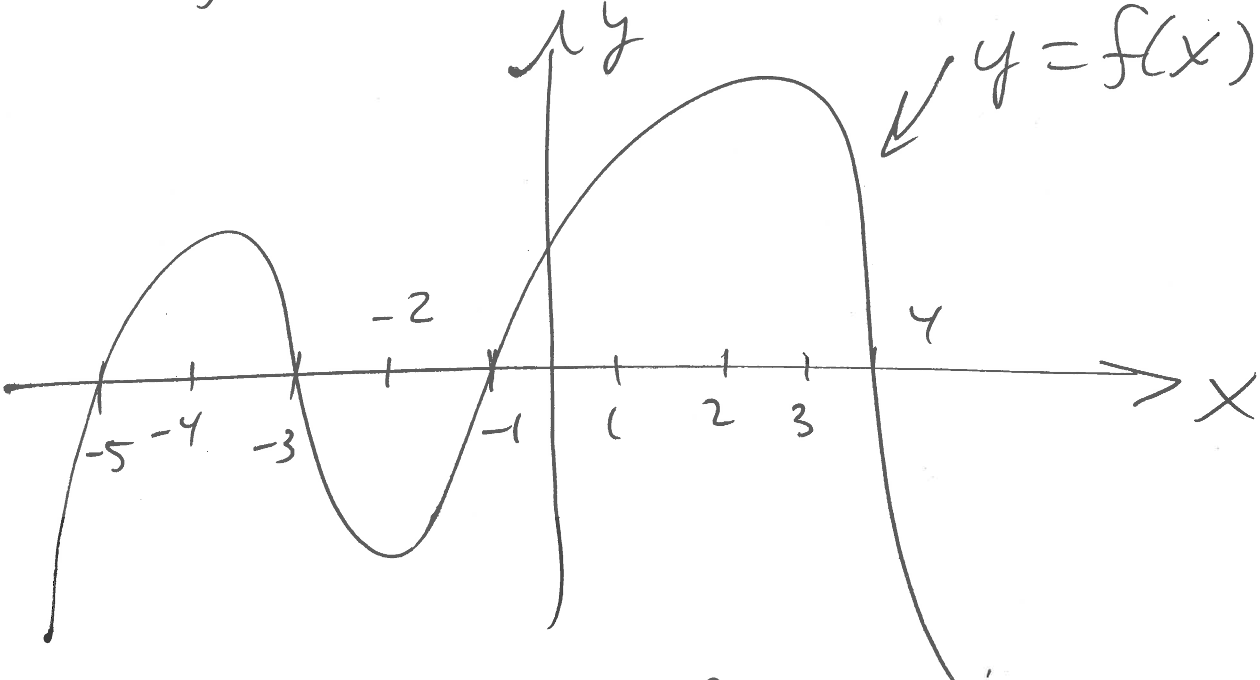


$$8 \geq 3x + 5 > -13$$
$$-5 \quad -5 \quad -5$$
$$\frac{3}{3} \geq \frac{3x}{3} > \frac{-18}{3}$$
$$\boxed{1 \geq x > -6}$$

$\boxed{(-6, 1]}$

$y = \underbrace{\text{some polynomial}} = \underline{f(x)}$

$f(x) > 0$?



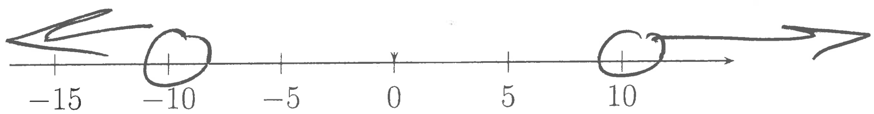
Answer to $f(x) > 0$ is range(s) of x-values

$$\boxed{(-5, -3) \cup (-1, 4)}$$

2.4.4 Absolute value and inequalities

Absolute value is still two equations

Example 2.4.5. $\left|\frac{x}{2}\right| > 5$

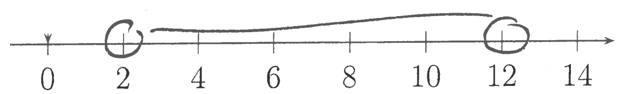


$2\left(\frac{x}{2} > 5\right)$ OR $\left(-\frac{x}{2} > 5\right)(-2)$

$\boxed{x > 10 \qquad x < -10}$

$\boxed{(-\infty, -10) \cup (10, \infty)}$

Example 2.4.6. $|x - 7| < 5$



$-5 < x - 7 < 5$

$\underline{+7 \quad +7 \quad +7}$

$2 < x < 12$

Question: What does $|x - 2| < 5$ mean?

Answer: All real numbers within five units of two. less than

So all real numbers within 5 units of 8 would be written as:

$|x - 8| < 5$

And all real numbers at least 5 units from 8 would be written as: greater than

$|x - 8| \geq 5$

2.4.5 Solving polynomial inequalities

Example 2.4.7. $x^2 < 5$

Step 1: Set equation equal to zero and find the zeros. (Factor)

$x^2 - 5 < 0$

x-intercepts

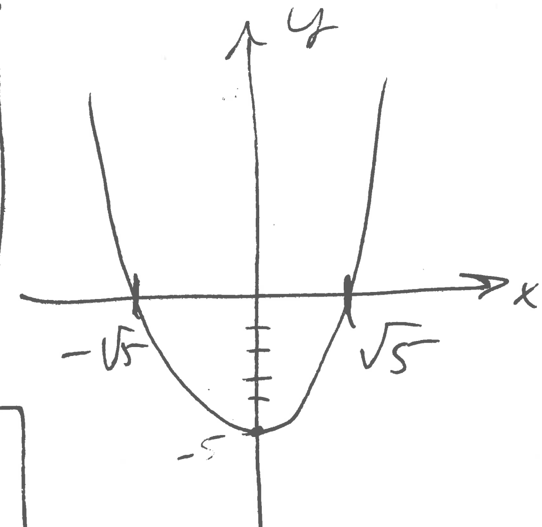
$x = \pm\sqrt{5}$

Step 2: Set up a table of signs

$(-\infty, -\sqrt{5}) \quad -\sqrt{5} \quad (-\sqrt{5}, \sqrt{5}) \quad \sqrt{5} \quad (\sqrt{5}, \infty)$

$x = -10 \qquad x = 0 \qquad x = 10000000$

$+ \qquad - \qquad +$



U-pick.

$$7 + 2|x+3| < 21$$
$$-7 \qquad\qquad -7$$

$$\frac{2|x+3|}{2} < \frac{14}{2}$$

$$|x+3| < 7$$

$$-7 < x+3 < 7$$

$$\cancel{\underbrace{7+2}_{\text{NO}}(x+3) = 21} \qquad \cancel{\text{PEMDAS}}$$

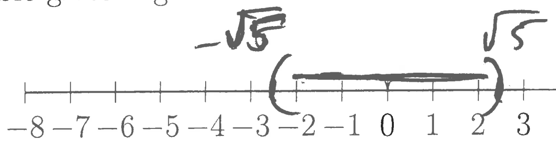
$$-7 < x+3 < 7$$
$$-3 \qquad -3 \qquad -3$$

$$\boxed{\begin{array}{l} -10 < x < 4 \\ (-10, 4) \end{array}}$$

try $x=4$ $\quad 7 + 2|4+3| = 7 + 2(7)$

$$= 7 + 14$$
$$= 21$$

Step 3: Find where the table gives negative values and write the solution



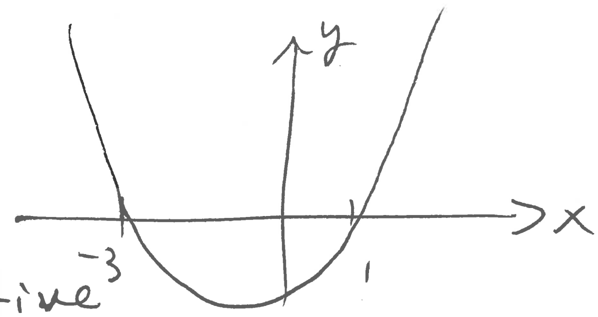
$\boxed{(-\sqrt{5}, \sqrt{5})}$

Example 2.4.8. $x^2 + 2x - 3 \geq 0$

$(x+3)(x-1) = 0 \Rightarrow x = -3, 1$

	-3			1	
-5		0		20	← pick x
$(-)(-)$		$(+)(-)$		$(+)(+)$	
$+$		$-$		$+$	

$\boxed{(-\infty, -3] \cup [1, \infty)}$



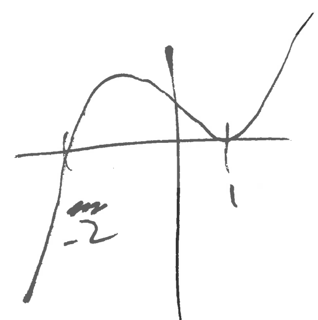
Example 2.4.9. $(x-1)^2(x+2)^3 \geq 0$ positive

Where is it zero?

$(x-1)^2(x+2)^3 = 0$

$(x-1)^2 = 0$ or $(x+2)^3 = 0$

$x = 1$ $\qquad$ $x = -2$



$\boxed{[-2, \infty)}$

	-2		1	
Pick x	-5	0		10
	$(+)(-)$	$(+)(+)$		$(+)(+)$
	$-$	$+$		$+$