Contents

2.1 The Graph of a Quadratic Equation ..................................... 2
2.2 Polynomial Functions of Higher Degree ................................ 5
2.3 Polynomial and Synthetic Division ....................................... 6
2.4 Complex Numbers ............................................................ 10
2.5 Fundamental Theorem of Algebra ....................................... 12
2.6 Rational Functions ........................................................... 14
2.1 The Graph of a Quadratic Equation

Definition 2.1. Let \(a, b,\) and \(c\) be real numbers with \(a \neq 0\). The function

\[ f(x) = ax^2 + bx + c \]

is called a quadratic equation.

The graph of a quadratic equation is a parabola. All parabolas are symmetric with respect to the axis of symmetry which passes through the vertex.

Example 2.1.1.

\[ f(x) = x^2 + 2x + 1 \quad \text{OR} \quad f(x) = -x^2 - 2x - 1 \]

\(a > 0\) graph opens up \(a < 0\) graph opens down

The Standard Form of a Parabola

\[ f(x) = a(x - h)^2 + k \]

Vertex is located at \((h, k)\)

if \(a > 0\) graph opens up
if \(a < 0\) graph opens down

Alternate form

If

\[ f(x) = ax^2 + bx + c \]

then the

\[ \text{vertex} = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right). \]
To Graph a Parabola

1. Find the vertex
2. Find the $x$ - intercepts
3. Determine if it opens up or down
4. Sketch

Example 2.1.2. Graph $f(x) = 2x^2 - 12x - 14$.

**Step 1:** Vertex:

$x = \frac{-(-12)}{2(2)} = 3$

$y = f(3) = 2(9) - 12(3) - 14 = -32$

So the coordinates of the vertex are $(x, y) = (3, -32)$

**Step 2:** $x$ - intercepts:

$0 = 2x^2 - 12x - 14$

**Step 4:** Sketch
Example 2.1.3. Graph \( f(x) = (x - 6)^2 + 3 \)

Step 4: Sketch

Example 2.1.4. Find the standard form for a parabola that has \((0,1)\) as its vertex and passes through the point \((1,0)\)
2.2  Polynomial Functions of Higher Degree

For polynomial functions of higher degree than 2 we will primarily be concerned with finding zeros. The best way to find zeros is to FACTOR.

Example 2.2.1. Find all zeros of \( f(x) = x^4 - x^3 - 20x^2 \)

To find zeros you must set the function equal to zero. Do not do this unless you are looking for zeros.

\[
\begin{align*}
x^4 - x^3 - 20x^2 &= 0 \\
x^2(x^2 - x - 20) &= 0 \\
x^2(x - 5)(x + 4) &= 0
\end{align*}
\]

Now we have three things multiplied together that equal zero so one of them must be zero.

\[
x^2 = 0 \quad \text{OR} \quad x - 5 = 0 \quad \text{OR} \quad x + 4 = 0.
\]

Example 2.2.2. Find the zeros of the following: \( f(x) = 49 - x^2 \) and \( f(x) = 2x^4 - 2x^2 - 40 \)

If you know the zeros of the polynomial you can easily write the polynomial.

Example 2.2.3. Find a polynomial with the following zeros: \( x = -2, 0, 6 \).
2.3 Polynomials and Synthetic Division

We can use long division to divide polynomials the same way we use long division to divide integers.

Example 2.3.1. Simplify this fraction: \[ \frac{x^4 + 5x^3 + 6x^2 - x - 2}{x + 2} \]

So we know that

\[ \frac{x^4 + 5x^3 + 6x^2 - x - 2}{x + 2} = x^3 + 3x^2 - 1. \]

More specifically we have a factor of \( f(x) = x^4 + 5x^3 + 6x^2 - x - 2 \).

\[ x^4 + 5x^3 + 6x^2 - x - 2 = (x^3 + 3x^2 - 1)(x + 2). \]
Example 2.3.2. \[
\frac{5x^3 + 18x^2 + 8x - 6}{x + 3}
\]

So we know that
\[
\frac{5x^3 + 18x^2 + 8x - 6}{x + 3} = 5x^2 + 3x - 1 + \frac{-3}{x + 3}.
\]

If we write it with no denominators we get an expression of the form.
\[
5x^3 + 18x^2 + 8x - 6 = (5x^2 + 3x - 1)(x + 3) - 3
\]

This form illustrates the theorem known as **The Division Algorithm** which states that any two polynomials \(P(x)\) and \(D(x)\) (where \(D(x)\) is of lower degree) can be written as
\[
P(x) = D(x) \cdot q(x) + r(x)
\]

More formally:

**The Division Algorithm**

If \(P(x)\) and \(D(x)\) are polynomials such that \(D(x) \neq 0\) and the degree of \(D(x)\) is less than or equal to the degree of \(P(x)\), there exist unique polynomials \(q(x)\) and \(r(x)\) such that
\[
P(x) = D(x) \cdot q(x) + r(x)
\]

OR
\[
\frac{P(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}
\]
**Synthetic Division** (The easy way to divide)

**Example 2.3.3.** Divide using synthetic division: \[ \frac{5x^3 + 18x^2 + 8x - 6}{x + 3} \]

**Example 2.3.4.** Suppose we know that \( x = -2 \) is a zero of the polynomial \( f(x) = 2x^3 + x^2 - 5x + 2 \). Find all the zeros of the polynomial.

Since we know that \( x = -2 \) is a zero then we know that

\[ f(x) = 2x^3 + x^2 - 5x + 2 = (x + 2) \cdot q(x) \]

and we can use synthetic division to find \( q(x) \).
Example 2.3.5. Use synthetic division to divide \( \frac{-3x^4}{x+2} \)

Example 2.3.6. Given that \((x + 2)\) and \((x - 4)\) are factors of \(f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24\) find all the zeros of \(f(x)\).
2.4 Complex Numbers

Q: What is a complex number?

A: It is a number of the form $a + bi$ where $a$ and $b$ are real numbers and $i^2 = -1$.

- $a$ is called the **real part**
- $b$ is called the **imaginary part**.
- The **conjugate** of $a + bi$ is $a - bi$.

Properties

1. $a + b i = c + d i \iff a = c$ and $b = d$
2. $(a + b i) + (c + d i) = (a + c) + (b + d) i$
3. $(a + b i) \cdot (c + d i) = ac + ad i + bc i + bd i^2 = (ac - bd) + (ad + bc) i$
4. $(a + b i)(a - b i) = a^2 + b^2$

Example 2.4.1.

a. $(4 + i) + (5 + 3i)$

b. $(2i + 7) - 2i$

c. $(3 + 2i) + (4 - i) - (7 + i)$

d. $(5 + 2i)(4 - 3i)$

e. $\frac{i}{3 + i}$
f. \[
\frac{3 - 5i}{2 - i}
\]

g. \((3 - \sqrt{-4}) + (-8 + \sqrt{-25})\)

h. \((\sqrt{-5})(\sqrt{-5})\)

i. \((2 - \sqrt{-1})(5 + \sqrt{-9})\)

j. \[
\frac{1}{3i}
\]

Complex numbers in quadratic equations

Example 2.4.2. Solve for \(x\): \(x^2 + 6x + 10 = 0\)

Example 2.4.3. Solve for \(x\): \(9x^2 - 6x + 37 = 0\)

Example 2.4.4. Solve for \(x\): \(x^2 + 1 = 0\)
2.5 Fundamental Theorem of Algebra

The two main ideas:

**The Linear Factorization Theorem**
If \( f(x) \) is a polynomials of degree \( n \), where \( n > 0 \), then \( f \) has precisely \( n \) linear factors

\[
 f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)
\]

where \( c_1, c_2, \ldots, c_n \) are complex numbers.

**The Rational Roots Test**
If the polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \) has integer coefficients, every rational zero of \( f \) has the form

\[
 \text{Rational zero} = \frac{p}{q}
\]

where \( p \) and \( q \) have no common factors other than 1, and

\[ p = \text{a factor of the constant term } a_0 \]
\[ q = \text{a factor of the constant term } a_n. \]

**Example 2.5.1.** Find all the zeros of

1. \( f(x) = (x + 5)(x - 8)^2 \)

2. \( f(t) = (t - 3)(t - 2)(t - 3i)(t + 3i) \)

**Example 2.5.2.** Find a polynomial function with integer coefficients that has the zeros

1. \( 4, 3i, -3i \).

2. \( -5, 1 + \sqrt{3}i \)
Example 2.5.3. Use the rational roots test to find all the real zeros of

1. \( g(x) = x^3 - 4x^2 - x + 4 \)

2. \( p(x) = x^3 - 9x^2 + 27x - 27 \)

Example 2.5.4. Find all the roots of \( f(x) = x^3 - 7x^2 - x + 87 \) knowing that \( 5 + 2i \) is a zero.
2.6 Rational Functions

Q: What is a rational function?
A: It is a function of the form:

\[ f(x) = \frac{N(x)}{D(x)} \quad D(x) \neq 0. \]

Rational functions can have Vertical and Horizontal Asymptotes

A Vertical Asymptote describes the behavior of a function near a discontinuity. They occur at any \( x \)-value where the numerator IS NOT equal to zero but the denominator IS equal to zero.

Example 2.6.1. Find vertical asymptotes for

\[ f(x) = \frac{1}{x} \quad \text{and} \quad f(x) = \frac{1}{x-3} \quad \text{and} \quad f(x) = \frac{1-3x}{x(x-3)}. \]

A Horizontal Asymptote describes the behavior of a function as \( x \) gets very large.

(ie. What happens to \( y \) as \( x \) goes to \( \infty \)?)

**Horizontal Asymptotes**

Let \( f \) be the rational function given by

\[ f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0} \]

where \( N(x) \) and \( D(x) \) have no common factors. The graph of \( f \) has one or no horizontal asymptote determined by comparing the degrees of \( n(x) \) and \( D(x) \).

1. If \( n < m \), then the graph of \( f \) has the line \( y = 0 \) (the \( x \)-axis) as a horizontal asymptote.
2. If \( n = m \) then the graph of \( f \) has the line \( y = \frac{a_n}{b_m} \) as a horizontal asymptote.
3. If \( n > m \) then the graph of \( f \) has no horizontal asymptote.
Example 2.6.2. Find the domain of the function and identify any horizontal and vertical asymptotes. Sketch a graph for 1, 2, 4, 5 and 6.

1. \( f(x) = \frac{4}{(x - 2)^2} \)
2. \( f(x) = \frac{1 - 5x}{1 + 2x} \)
3. \( f(x) = \frac{2x^2}{x + 1} \)
4. \( f(x) = \frac{1}{x - 3} \)
5. \( f(t) = \frac{1 - 2t}{t} \)
6. \( f(x) = \frac{2x}{x^2 + x - 2} \)