11 Probability

11.1 Basic Concepts

Start with some Definitions:

Experiment: Any observation of measurement of a random phenomenon is an experiment.

Outcomes: Any result of an experiment is called an outcome.

Sample Space: The set of all possible outcomes is called the sample space.

Theoretical Probability Formula

If all the outcomes in a sample space \( S \) are equally likely, and \( E \) is an event within that sample space, then the theoretical probability of the event \( E \) is given by

\[
P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.
\]

Empirical Probability Formula

If \( E \) is an event that may happen when an experiment is performed, then the empirical probability of the event \( E \) is given by

\[
P(E) = \frac{\text{number of times event } E \text{ occurred}}{\text{number of times the experiment was performed}}.
\]

Law of Large Numbers

As an experiment is repeated more and more times, the proportion of outcomes favorable to any particular event will tend to come closer and closer to the theoretical probability of that event.

Odds

If all outcomes in a sample space are equally likely, \( a \) of them are favorable to the event \( E \), and the remaining \( b \) outcomes are unfavorable to \( E \), then the odds in favor of \( E \) are \( a \) to \( b \), and the odds against \( E \) are \( b \) to \( a \).

Example 1. A dart lands at random on the board shown. Find the probability that the dart will land on blue. On red? On yellow?

\[
\begin{array}{ccc}
R & Y \\
B & B & R & Y \\
B & Y \\
\end{array}
\]
Example 2. Five people (Andy, Bill Cathy, David, and Evelyn) form a club
\[ N = \{A, B, C, D, E\}. \] If they choose a president randomly, find the odds against each of the following results.

1. David
2. a woman
3. a person whose name begins with a vowel

Example 3. "‘Jukebox’' Joe has fifty hit singles from the fifties, including exactly one by Buddy Holly, two by The Drifters, three by Bobby Darin, four by The Coasters, and five by Fats Domino. If Joe randomly selects one hit from his collection of fifty, find the probability it will be by each of the following.

1. Buddy Holly
2. The Drifters
3. Bobby Darin
4. The Coasters
5. Fats Domino

Example 4. Three fair coins are tossed. Write out the sample space, then determine the probability of each of the following events.

1. no heads
2. exactly two heads
3. exactly one head
4. three heads
**Example 5.** In a certain state, 38,550 boys and 36,770 girls were born in 1996. Find the empirical probability that one of those births, chosen at random, would be a boy. A girl?

**Example 6.** A pair of dice is rolled. What is the probability of getting a 7 or an 11?

**Example 7.** If $P(E) = 0.33$, what are the odds against $E$?

**Example 8.** In 5-card poker the number of outcomes favorable to an event $E$ is given in the table. Find the probability of being dealt a full house.

<table>
<thead>
<tr>
<th>Event $E$</th>
<th># of Outcomes favorable to $E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Royal flush</td>
<td>4</td>
</tr>
<tr>
<td>Straight flush</td>
<td>36</td>
</tr>
<tr>
<td>Four of a kind</td>
<td>624</td>
</tr>
<tr>
<td>Full house</td>
<td>3744</td>
</tr>
<tr>
<td>Flush</td>
<td>5108</td>
</tr>
<tr>
<td>Straight</td>
<td>10,200</td>
</tr>
<tr>
<td>Three of a kind</td>
<td>54,912</td>
</tr>
<tr>
<td>Two pairs</td>
<td>123,552</td>
</tr>
<tr>
<td>One pair</td>
<td>1,098,240</td>
</tr>
<tr>
<td>No pair</td>
<td>1,302,540</td>
</tr>
<tr>
<td>Total</td>
<td>2,598,960</td>
</tr>
</tbody>
</table>
11.2 Events involving ”Not” and ”Or”

You may recall the Theoretical Probability formula from Section 11.1.

**Theoretical Probability Formula**

If all the outcomes in a sample space $S$ are equally likely, and $E$ is an event within that sample space, then the theoretical probability of the event $E$ is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.$$ 

Using this formula we can come up with some properties for the probability of an event $E$ occurring.

**Properties of Probability**

Let $E$ be an event within from sample space $S$. That is, $E$ is a subset of $S$. Then the following properties hold.

1. $0 \leq P(E) \leq 1$ (The probability of an event is a number from 0 through 1, inclusive.)
2. $P(\emptyset) = 0$ (The probability of an impossible event is 0.)
3. $P(S) = 1$ (The probability of a certain event is 1.)

**Complements Rule of Probability**

The probability that an event $E$ will occur is one minus the probability that it will not occur.

$$P(E) = 1 - P(E')$$

**General Addition Rule of Probability**

If $A$ and $B$ are any two events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B),$$

OR

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$
Example 1. For the experiment of rolling a single fair die, find the probability of each of the following events.

1. not less than 2
2. not prime
3. odd or less than 5
4. even or prime

Example 2. For the experiment of drawing a single card from a standard 52-card deck, find (a) the probability, and (b) the odds in favor, of each of the following events.

1. not an ace
2. king or a queen
3. club or a heart
4. not a heart or a seven

Example 3. For the experiment of rolling an ordinary pair of dice, find the probability that the sum will be each of the following.

1. 11 or 12
2. even or a multiple of 3
3. odd or greater than 9
4. less than 3 or greater than 9

Example 4. If you are dealt a 5-card hand from a standard 52-card deck, find the probability of getting each of the following. Refer to the table in Example 8 in Section 11.1, and give answers to six decimal places.

1. a flush or three of a kind
2. a full house or a straight
3. a black flush or two pairs
4. nothing any better than two pairs
Example 5. The table gives golfer Amy Donlin’s probabilities of scoring in various ranges on a par-70 course. In a given round find the probability that her score will be each of the following.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Below 60</td>
<td>.04</td>
</tr>
<tr>
<td>60-64</td>
<td>.06</td>
</tr>
<tr>
<td>65-69</td>
<td>.14</td>
</tr>
<tr>
<td>70-74</td>
<td>.30</td>
</tr>
<tr>
<td>75-79</td>
<td>.23</td>
</tr>
<tr>
<td>80-84</td>
<td>.09</td>
</tr>
<tr>
<td>85-89</td>
<td>.06</td>
</tr>
<tr>
<td>90-94</td>
<td>.04</td>
</tr>
<tr>
<td>95-99</td>
<td>.03</td>
</tr>
<tr>
<td>100 or above</td>
<td>.01</td>
</tr>
</tbody>
</table>

1. 95 or higher
2. par or above
3. in the 80’s
4. less than 90
5. What are the odds of shooting below par?

11.3 Events Involving "And"

Conditional Probability

The probability of event $B$, computed on the assumption that event $A$ has happened, is called the conditional probability of $B$, Given $A$, and is denoted $P(B \mid A)$.

General Multiplication Rule of Probability

If $A$ and $B$ are any two events, then

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A),$$

OR

$$P(A \cap B) = P(A) \cdot P(B \mid A).$$

Example 1. A pet store has seven puppies, including 4 poodles, 2 terriers, and 1 retriever. If Rebecka and Aaron, in that order, each select one puppy at random, with replacement (they may both select the same one), find the probability of the following events.

1. both select a poodle

2. Rebecka selects a retriever, Aaron selects a terrier

3. Rebecka selects a terrier, Aaron selects a retriever.
Example 2. Suppose the puppies are selected as in Example 1, but this time without replacement (they cannot both select the same puppy). Find the probability of the following events.

1. both select a poodle

2. Aaron selects a terrier, given Rebecka selects a poodle

3. Aaron selects a retriever, given Rebecka selects a poodle

4. both select a retriever

Example 3. Let two cards be dealt successively, without replacement, from a standard 52-card deck. Find the probability of each of the following events.

1. spade second, given spade first

2. club second given diamond first

3. two face cards

4. no face cards

Example 4. For a single event with two conditions the conditional probability can be calculated with the following formula \( P(B \mid A) = \frac{n(A \text{ and } B)}{n(A)} = \frac{n(A \cap B)}{n(A)} \). Find each probability when a single card is drawn from a standard 52 card deck.

1. \( P(\text{queen} \mid \text{face card}) \)

2. \( P(\text{face card} \mid \text{queen}) \)
**Independent Events**

Two events $A$ and $B$ are called independent if knowledge about the occurrence of one of them has no effect on the probability of the other one, that is, if

$$P(A \mid B) = P(B)$$

**Special Multiplication Rule of Probability**

If $A$ and $B$ are independent events, then

$$P(A \text{ and } B) = P(A) \cdot P(B),$$

OR

$$P(A \cap B) = P(A) \cdot P(B).$$

**Example 5.** Emily manages a grocery warehouse. Emily has discovered that on any given weekday, 80% of the customer sales amount to more than $100. Find the probability of each of the following events.

1. The first two sales on Wednesday are both for more than $100.

2. The first three sales on Wednesday are all for more than $100.

3. Exactly one of the first three sales on Wednesday is for more than $100.

**Example 6.** If it snows tomorrow, the probability is 0.5 that John will practice his guitar. If it does not snow tomorrow, there is only a 0.4 chance that John will practice. Suppose the chance of snow tomorrow is 60%. What is the probability that John will practice his guitar?
11.4 Binomial Probability

When the outcomes of an experiment are divided into just two categories, success and failure, the associated probabilities are called binomial. It is NOT necessary for the probability of success to be the same as the probability of failure. If you flip a coin then the probability of heads in 1/2 and the probability of tails is 1/2. If we were throwing a dart at a dartboard and we wanted to know whether we hit the number 20 or not this would be a binomial probability, either we hit 20 or we don’t. Notice that the probability of hitting 20 on a dart board is only 1/20 (probability of success) and for missing is 19/20 (probability of failure).

When we talk about binomial probability we often refer to $x$ as the number of successful outcomes and we talk about $P(x)$ which is the probability of having $x$ successful outcomes.

### Binomial Probability Formula

When $n$ independent repeated trials occur, where

\[ p = \text{probability of success} \quad \text{and} \quad q = \text{probability of failure} \]

with $p$ and $q (=1-p)$ remaining constant throughout all $n$ trials, the probability of exactly $x$ successes is given by

\[ P(x) = \binom{n}{x} p^x q^{n-x} \]

**Example 1.** Ten coins dated 1990 through 1999 are tossed. What is the probability of the following events.

A. $P(\text{heads on the 1990 coin only})$.

B. $P(\text{heads on all 10 coins})$

C. $P(\text{exactly 5 heads})$

D. $P(\text{at least three heads})$

**Example 2.** Assume the probability is 1/2 that a child born is a boy. Find the probability that a family’s fourth child will be their first daughter. If the family has three children what is the probability that they have a) exactly one boy? b) at most two girls?
Example 3. For $n$ repeated independent trials, with constant probability of success $p$ for all trials, find the probability of exactly $x$ successes in each of the following cases.

1. $n = 5, p = 1/3, x = 4$

2. $n = 10, p = .7, x = 5$

3. $n = 20, p = 1/8, x = 22$

Example 4. The probability of rolling a number greater than 2 on a die is $2/3$. Find the probability that on the next 8 rolls of a die exactly 6 show a number greater than 2.

Example 5. It is known that a certain prescription drug produces undesirable side effects in 30 percent of all patients who use it. Among a random sample of eight patients using the drug, find the probability of each of the following events.

1. None have undesirable side effects.

2. Exactly one has undesirable side effects.

3. Exactly two have undesirable side effects.

4. More than two have undesirable side effects.