6.7 Homogeneous Systems - Repeated Eigenvalues

Suppose: \( r \) is an eigenvalue of multiplicity two for a matrix \( A \), and \( E_r = \{ \mathbf{v} \} \), i.e. dimension one. The following are true:

1. \( \mathbf{v} \) is a non-trivial solution to the equation ______________________

2. \( \phi_1 = \bar{v} e^{rt} \) satisfies ______________________

Now, assume there is a second linearly independent solution of the form
\[
\phi_2(t) = ______________________ . \quad \text{This solution must satisfy ______________________} .
\]

Conclusion:

We need something else. Assume there is a solution of the form
\[
\phi_3(t) = ______________________ \quad \text{for some} \quad ______________________ . \quad \text{This solution}
\]

must satisfy ______________________ , and

\[
\phi_4(t) =
\]
so in the differential equation $\ddot{x} = A\dot{x}$, we have

So: $\phi_2(t) =$ \underline{solution} to \underline{equation} if \underline{solution} to \underline{equation}.

Conclusion: if $r$ is an eigenvalue of multiplicity two and dimension one and $E_r = \{ v \}$, then two linearly independent solutions are

$\phi_1 =$ \underline{solution}

and $\phi_2 =$ \underline{solution}

where

And the general solution is
Ex 1: \begin{align*} x' &= 2x + y \\ y' &= -x + 4y \\ X(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{align*}

First write this as a matrix equation:

\[ X' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} X \]

Solve for the eigenvalues:

\[ \begin{pmatrix} 2 - r & 1 \\ -1 & 4 - r \end{pmatrix} = (2 - r)(4 - r) + 1 = r^2 - 6r + 9 = 0 \]

So \( r = 3 \) and 3

Solve for the eigenvectors:

\[ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x = y \]

So the eigenvector is \( \beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

Our first choice for a solution is \( X^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \)

The second choice for a solution is \( X^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{3t} + \eta e^{3t} \) where \( \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \) satisfies the equation:

\[ (A - rI) \eta = \beta \]

\[ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
Ex 2: $x' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} x$ \hspace{1cm} x(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
Ex 3:  
\[ x^s = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} x \]