6.4 Fundamental Sets and Linear Independence

Consider the following system of differential equations:

\[
y' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}y
\]

Two solutions are

\[
y_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \quad \text{and} \quad y_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}
\]

We can write these solutions as one matrix.

\[
\psi(t) = \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix}
\]

The Wronskian of this set is \( W(t) = \det(\psi(t)) = -e^{-t} e^{-3t} - e^{-t} e^{-3t} = -2e^{-4t} \neq 0 \). This brings us to

**Theorem 6.6:** Let \( \{ y_1(t), y_2(t), \ldots, y_n(t) \} \) be a fundamental set of solutions of

\[
y' = P(t)y, \quad a < t < b,
\]

where the \((nxn)\) matrix function \( P(t) \) is continuous on the interval \((a, b)\). Let \( W(t) \) denote the Wronskian of the solution set and let \( t_0 \) be any point in the interval \((a, b)\). Then \( W(t_0) \) is nonzero.

Q. What does it mean to be a fundamental set of solutions?
A. It means that the general solution to the system of equations can be written:

\[
y(t) = \psi(t)c \quad \text{where} \quad c = [c_1, c_2, \ldots, c_n]
\]

Fundamental sets are always linearly independent.

Q. How are fundamental sets of solutions related?
A. Theorem 6.8:

Let \( \{ y_1(t), y_2(t), \ldots, y_n(t) \} \) be a fundamental set of solutions of

\[
y' = P(t)y, \quad a < t < b,
\]

where the \((nxn)\) matrix function \( P(t) \) is continuous on the interval \((a, b)\). Let

\[
\Psi(t) = \begin{bmatrix} y_1(t), y_2(t), \ldots, y_n(t) \end{bmatrix}
\]

denote the \(nxn\) matrix function formed from the fundamental set. Let \( \{ \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_n(t) \} \) be any other set of \(n\) solutions of the differential equation, and let
\[ \bar{\Psi}(t) = [\bar{y}_1(t), \bar{y}_2(t), \ldots, \bar{y}_n(t)] \]
denote the \( nxn \) matrix function formed from this other set of solutions. Then

(a) there is a unique \( nxn \) constant matrix \( C \) such that
\[ \bar{\Psi}(t) = \psi(t)C \]
(b) Moreover, \( \{ \bar{y}_1(t), \bar{y}_2(t), \ldots, \bar{y}_n(t) \} \) is also a fundamental set of solutions if and only if the determinant of \( C \) is nonzero.

Ex 1: Determine whether the given functions are linearly independent or linearly dependent.

a) \[ \begin{bmatrix} e^t \\ 1 \end{bmatrix}, \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} \]

b) \[ \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix} \]

Ex 2: a) Verify that the matrix \( \psi(t) \) is a fundamental matrix of the given linear system.

b) Determine a constant matrix \( C \) such that the given matrix \( \bar{\psi}(t) \) can be represented as \( \bar{\Psi}(t) = \psi(t)C \).

c) Use your knowledge of the matrix \( C \) and assertion (b) of theorem 6.8 to determine whether \( \bar{\psi}(t) \) is also a fundamental matrix or simply a solution matrix.

I) \[ y' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y, \quad \psi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}, \quad \bar{\psi}(t) = \begin{bmatrix} \sinh(t) & \cosh(t) \\ \cosh(t) & \sinh(t) \end{bmatrix} \]

II) \[ y' = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} y, \quad \psi(t) = \begin{bmatrix} e^t & e^{-t} & 4e^{2t} \\ e^t & -2e^{-t} & e^{2t} \\ 0 & 0 & 3e^{2t} \end{bmatrix}, \quad \bar{\psi}(t) = \begin{bmatrix} e^t + e^{-t} & 4e^{2t} & e^t + 4e^{2t} \\ -2e^{-t} & e^{2t} & e^{2t} \\ 0 & 3e^{2t} & 3e^{2t} \end{bmatrix} \]