Section 9.7 Inverse Trigonometric Functions

Recall: If we have a function \( y = f(x) \) then the inverse function \( y = f^{-1}(x) \) is that function which has the property: \( f^{-1}(f(x)) = f(f^{-1}(x)) = x \). In some sense it undoes what the original function had done. One important property of the inverse function is that it is a function, in other words for every \( x \) you put into the function you get out exactly one \( y \) value.

We already learned how to solve for the inverse of a function in precalculus and there are two ways to do it. The easiest thing to do is to look at the graph and reflect it across the line \( y = x \). The harder way to solve for the inverse of a function is to switch the \( x \) and \( y \) coordinates and solve for \( y \).

We can find inverse functions of the sine and cosine and we will use the graphing method.

The first thing we need to do is to restrict the domain so that when we reflect our function across the line \( y = x \) we end up with a function. Consider the basic graph of sine (Figure 1):

![Figure 1](image.png)

The sine takes on \( y \) values in the range: \(-1 \leq y \leq 1\) so when we find the inverse function the domain must be the same. The area highlighted in red covers the appropriate span \(-\pi/2 \leq x \leq \pi/2\).

Now all we need to do is to reflect this red area across the line \( y = x \) to get the inverse sine function. This function is called the arcsine or \( y = \arcsin x \). The graph of the arcsine is in Figure 2 at the top of the next page.

Note: With the restriction we have put on the arcsine, it is ONLY defined in quadrants I and IV so all your answers for arcsine must lie between \(-\pi/2 \) and \( \pi/2 \).
The cosine is similar but in this case we will restrict the domain to \(0 \leq x \leq \pi\) because this also gives us all the \(y\) values between \(-1\) and \(1\). This is shown in the highlighted red area of the cosine function (see Figure 3).

Again we reflect this red area across the line \(y = x\) to get the inverse cosine function. This function is called the arcsine or \(y = \arccos x\). We can see this in the next graph (Figure 4).

**Note:** With the restriction we have put on the arccosine, it is ONLY defined in quadrants I and II so all your answers for arccosine must lie between 0 and \(\pi\).
Figure 4

For all of these inverse functions you can read them the following way:

\[ y = \arcsin x : \]  
\[ y \] is the angle whose sine is \( x \).  
In other words \( \sin x \) gives you a ratio and \( \arcsin x \) gives you an angle.

\[ y = \arccos x : \]  
\[ y \] is the angle whose cosine is \( x \).  
In other words \( \cos x \) gives you a ratio and \( \arccos x \) gives you an angle.

\[ y = \arctan x : \]  
\[ y \] is the angle whose tangent is \( x \).  
In other words \( \tan x \) gives you a ratio and \( \arctan x \) gives you an angle.

Note: The arctangent is only defined from \(-\pi/2 \leq x \leq \pi/2\).

On your calculator these functions are not displayed as arc functions. Writing arccosine and arcsine takes up very much space so we most often use a short hand version of the inverse:

\[ \arcsin x = \sin^{-1} x \]

and

\[ \arccos x = \cos^{-1} x \]

These two notations will be used interchangeable in this class and the second form is used by your calculator. \( \sin^{-1} x \) is often found just above \( \sin x \) but different models of calculator have it in different places.
Ex 1: \( y = \arccos \left( \frac{\sqrt{3}}{2} \right) \). Find \( y \).

Step 1: Draw a triangle in the appropriate quadrant and label the sides. \( \arccos \) is defined in quadrant I and II and \( \frac{\sqrt{3}}{2} \) is positive so we draw the triangle in quadrant I (see Figure 5).

Step 2: Identify the angle in the triangle. Very often it will be one of the special triangles. In this case we have a 30°–60°–90 triangle so our angle is 30° or \( \pi/6 \).

![Figure 5](image)

Ex 2: \( y = \arcsin \left( \frac{\sqrt{2}}{2} \right) \). Find \( y \).
Ex 3.  \( y = \arcsin 0 \)

Ex 4.  \( y = \arctan(-1) \)

Ex 5.  \( \tan(\arctan(25)) = \)

Ex 6.  \( \cos \left( \cos \left( \frac{7\pi}{2} \right) \right) = \)

Ex 7.  \( \csc \left[ \tan^{-1} \left( -\frac{5}{12} \right) \right] = \)

Ex 8.  \( \tan \left( \sin^{-1} \left( -\frac{3}{4} \right) \right) \)