Section 9.4 Trigonometric Functions of any Angle

So far we have only really looked at trigonometric functions of acute (less than 90°) angles. We would like to be able to find the trigonometric functions of any angle.

To do this follow these steps:

1. Draw the angle in standard position on the coordinate axes
2. Draw a reference triangle and find the reference angle
3. Label the reference triangle
4. Write down the answer

OR use your calculator.

Note: Your calculator will only give you decimal approximations but on the test your answers must be exact.

What is a reference triangle and reference angle?

If you draw the angle \( \theta \) in the standard position its reference angle is the acute angle \( \theta' \) formed by the terminal side of \( \theta \) and the horizontal axis.

The reference triangle is the triangle which is formed by drawing a perpendicular line from the terminal side of your angle \( \theta \) in standard position to the horizontal axis.

Ex: Here is a reference angle and triangle in the 2nd quadrant.

The size of the reference angle in the second quadrant will be \( 180 - \theta \) or \( \pi - \theta \) depending on whether the angle is given in degrees or radians respectively.
Here is one in the 4th quadrant:

Reference Angle \( \theta' \)  
| Original Angle \( \theta \)  
\hline
\( \theta \)  
\hline
\( \theta' \)  
\hline
Reference Triangle

The size of the reference angle in the fourth quadrant will be \( 360 - \theta \) or \( 2\pi - \theta \) depending on whether the angle is given in degrees or radians respectively.

Ex 1: Find the reference triangles and angles:

a) \( \theta = 309^\circ \)
   
The reference angle will be \( 360 - 309 = 51^\circ \)
b) \[ \theta = -\frac{7\pi}{4} \]

The reference angle will be

\[ 2\pi - \frac{7\pi}{4} = \frac{\pi}{4} \]

\[ \theta' = \frac{\pi}{4} \]

\[ \theta \]

\[ \theta' \]

\[ \text{Reference Triangle} \]

\[ \text{Reference Angle } \theta' \]

\[ \text{Original Angle } \theta \]

\[ \frac{\pi}{3} \]

\[ \frac{4\pi}{3} \]

\[ \text{Reference Triangle} \]

\[ \text{Reference Angle } \theta' \]

\[ \text{Original Angle } \theta \]

\[ \frac{\pi}{3} \]

\[ \frac{4\pi}{3} \]

\[ \text{Reference Triangle} \]

\[ \text{Reference Angle } \theta' \]

\[ \text{Original Angle } \theta \]

\[ \frac{\pi}{3} \]

\[ \frac{4\pi}{3} \]

\[ \text{Reference Triangle} \]
Now we will use these reference angles to find the values of some trigonometric functions.

Ex 2. Find the values of the six trigonometric functions for $\theta = 150^\circ$

1. Draw the angle in standard position
2. Draw the reference triangle and angle
3. Label the triangle. Here we will label using the standard 30 – 60 - 90 triangle

NOTE: Since one side of the reference triangle is on the negative $x$-axis that side is labeled as $-\sqrt{3}$. This is VERY IMPORTANT. You will notice that this makes the cosine, secant, tangent and cotangent negative.

4. Now we can find the 6 trigonometric functions by reading them off the reference triangle:

$$\sin(150) = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$$

$$\csc(150) = \frac{\text{hyp}}{\text{opp}} = \frac{2}{1} = 2$$

$$\cos(150) = \frac{\text{adj}}{\text{hyp}} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$\sec(150) = \frac{\text{hyp}}{\text{adj}} = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}}$$

$$\tan(150) = \frac{\text{opp}}{\text{adj}} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\cot(150) = \frac{\text{adj}}{\text{opp}} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$
Ex 3: Find the values of the six trigonometric functions for \( \theta = -\frac{\pi}{4} \)

Note that the angle is negative.

1. Draw the angle in standard position
2. Draw the reference triangle and angle
3. Label the triangle. Here we will label using the standard 30 – 60 - 90 triangle

\[
\begin{align*}
\sin\left(-\frac{\pi}{4}\right) &= \frac{\text{opp}}{\text{hyp}} = \frac{-1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \\
\csc\left(-\frac{\pi}{4}\right) &= \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{2}}{-1} = -\sqrt{2} \\
\cos\left(-\frac{\pi}{4}\right) &= \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} \\
\sec\left(-\frac{\pi}{4}\right) &= \frac{\text{hyp}}{\text{adj}} = \sqrt{2} \\
\tan\left(-\frac{\pi}{4}\right) &= \frac{\text{opp}}{\text{adj}} = \frac{-1}{1} = -1 \\
\cot(150) &= \frac{\text{adj}}{\text{opp}} = \frac{1}{-1} = -1
\end{align*}
\]

**NOTE:** Since one side of the reference triangle is in the negative \( y \) direction that side is labeled as -1. This is VERY IMPORTANT. You will notice that this makes the sine, cosecant, tangent and cotangent negative.

4. Now we can find the 6 trigonometric functions by reading them off the reference triangle:
Ex 4: Suppose \( \cos \theta = -\frac{4}{5} \) and \( \theta \) in quadrant III. Find the values of the six trigonometric functions for \( \theta \).

The first thing we need to do is to draw a reference triangle. We know we are in quadrant III and we know that \( \cos \theta = -\frac{4}{5} = \frac{adj}{opp} \) so we know two sides of the triangle already.

We will use the Pythagorean theorem to find the last side. Here \( \theta' \) is the reference angle.

Solve for \( y \): \((-4)^2 + y^2 = 5^2 \) so \( y = 3 \).

**NOTE:** Since one side of the reference triangle is in the negative \( y \) direction and the other is in the negative \( x \) direction that both of those sides are negative. This is VERY IMPORTANT. You will notice that this makes the sine, cosecant, cosine and secant negative.

Now we can find the 6 trigonometric functions by reading them off the reference triangle:

\[
\begin{align*}
\sin(\theta) &= \frac{opp}{hyp} = -\frac{3}{5} = -\frac{3}{5} \\
csc(\theta) &= \frac{1}{\sin(\theta)} = -\frac{5}{3} = -\frac{5}{3} \\
\cos(\theta) &= \frac{adj}{hyp} = -\frac{4}{5} = -\frac{4}{5} \\
\sec(\theta) &= \frac{1}{\cos(\theta)} = -\frac{5}{4} = -\frac{5}{4} \\
\tan(\theta) &= \frac{opp}{adj} = -\frac{3}{-4} = \frac{3}{4} \\
cot(\theta) &= \frac{1}{\tan(\theta)} = -\frac{4}{3} = -\frac{4}{3}
\end{align*}
\]
What this has shown us is that we can determine the sign of the trigonometric functions by which quadrant the terminal side in.

<table>
<thead>
<tr>
<th>Quadrant II</th>
<th>Quadrant I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(\theta) ): +</td>
<td>( \sin(\theta) ): +</td>
</tr>
<tr>
<td>( \cos(\theta) ): -</td>
<td>( \cos(\theta) ): +</td>
</tr>
<tr>
<td>( \tan(\theta) ): -</td>
<td>( \tan(\theta) ): +</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quadrant III</th>
<th>Quadrant IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(\theta) ): -</td>
<td>( \sin(\theta) ): -</td>
</tr>
<tr>
<td>( \cos(\theta) ): -</td>
<td>( \cos(\theta) ): +</td>
</tr>
<tr>
<td>( \tan(\theta) ): +</td>
<td>( \tan(\theta) ): -</td>
</tr>
</tbody>
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Ex 5: Suppose \( \csc \theta = 4 \) and \( \cot \theta > 0 \). Find the values of the six trigonometric functions for \( \theta \)

Since the \( \frac{1}{\sin \theta} = \csc \theta = 4 \) the sine is positive so \( \theta \) is in quadrant I or II.

Since the \( \cot \theta > 0 \) then the tangent is positive so \( \theta \) is in quadrants I or III.

The overlap of these two regions is quadrant I so we can draw our triangle:

\[
\csc \theta = \frac{4}{1} = \frac{\text{hyp}}{\text{opp}}
\]

To solve for \( x \) we use the Pythagorean theorem: \( (1)^2 + x^2 = 4^2 \) so \( x = \sqrt{15} \).

\[
\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{1}{4} \quad \csc(\theta) = \frac{1}{\sin(\theta)} = 4
\]

\[
\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{15}}{4} \quad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{4}{\sqrt{15}}
\]

\[
\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{15}} \quad \cot(\theta) = \frac{1}{\tan(\theta)} = \sqrt{15}
\]