Section 4.3: The Product and Quotient Rules

**Product Rule:** If $f$ and $g$ are both differentiable, then

$$
\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [f(x)]
$$

It is often easier to think of this rule in terms of words: “The first times the derivative of the second plus the second times the derivative of the first.”

**Ex 1:**

a) $s(t) = (t^5 - 3t^2 + t)(t^4 - 3t^3 + 2t^2 - t)$

$$
s'(t) = (t^5 - 3t^2 + t)(4t^3 - 9t^2 + 4t - 1) + (t^4 - 3t^3 + 2t^2 - t)(5t^4 - 6t + 1)
$$

b) $f(x) = (x^3 + 2x^2 + \sqrt{x})(x^2 + \frac{1}{x})$

Rewrite this first: $f(x) = (x^3 + 2x^2 + x^{1/2})(x^2 + x^{-1})$

Now take a derivative: $f'(x) = (x^3 + 2x^2 + \sqrt{x})(2x - x^{-2}) + (x^2 + x^{-1})(3x^2 + 4x + \frac{1}{2}x^{-1/2})$

**Quotient Rule:** If $f$ and $g$ are both differentiable, then

$$
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}
$$

**NOTE:** There is a negative sign in this equation and yes it does matter which way you subtract.

This one is also easier to remember in words: “The bottom times the derivative of the top minus the top times the derivative of the bottom all divided by the denominator squared.”
Ex 2:

a) \( f(t) = \frac{t^2}{t^3 + 1} = \frac{\text{top}}{\text{bottom}} \)

\[
f'(t) = \frac{(t^3 + 1)^2(2t) - (t^3)(3t^2)}{(t^3 + 1)^2} = \frac{2t - 3t^2}{(t^3 + 1)^2}
\]

b) \( g(x) = \frac{\sqrt{x}}{x + 1} \)

Rewrite this first. Need exponents that are numbers

\[
g(x) = \frac{x^{1/2}}{x + 1}
\]

\[
g'(x) = \frac{(x + 1)^2 \left( \frac{1}{2} x^{-1/2} \right) - x^{1/2}}{(x + 1)^2}
\]

c) \( f(x) = \frac{2x + 1}{x^3 + 5x - 8} \)

\[
f'(x) = \frac{(x^3 + 5x - 8)(2) - (2x + 1)(3x^2 + 5)}{(x^3 + 5x - 8)^2}
\]

d) \( g(t) = \frac{1}{5t^2} \)

\[
g'(t) = \frac{5t^2(0) - (1)(10t)}{(5t^2)^2} = \frac{-10t}{25t^4} = \frac{-2}{5t^3}
\]
Ex 3:

a) \( f(x) = \left( 4 + \frac{1}{x} \right) \left( 2x - \frac{1}{x^2} \right) \)

b) \( h(x) = \frac{(\sqrt{x} + 3)}{\sqrt[3]{x}} \)

Rewrite first: \( h(x) = \frac{(x^{1/2} + 3)}{x^{1/3}} \)

c) \( g(t) = \frac{-5t^3 + 8t + -9 + \sqrt{t}}{t} \)
Higher Order Derivatives:

We can find the derivative of the derivative, this is known as the second derivative. This tells us the slope of the derivative. Similarly we can find the derivative of the second derivative, this is called the third derivative.

Notation:

Original function: \( y = f(x) \)

First derivative: \( y', \ f'(x), \frac{dy}{dx}, \frac{d}{dx}[f(x)] \)

Second derivative: \( y'', \ f''(x), \frac{d^2y}{dx^2}, \frac{d^2}{dx^2}[f(x)] \)

Third derivative: \( y''', \ f'''(x), \frac{d^3y}{dx^3}, \frac{d^3}{dx^3}[f(x)] \)

Fourth derivative: \( y^{(4)}, \ f^{(4)}(x), \frac{d^4y}{dx^4}, \frac{d^4}{dx^4}[f(x)] \)

You will notice that when we get to the fourth derivative we stop using the ‘ notation. The order of the derivative is written as \( (n) \) in the exponent.

Ex: Find the second and third derivatives of the following:

a) \( y = x + 8x^{-2} \)

Need the first derivative:

\[ y' = 1 - 16x^{-3} \]

The second derivative:

\[ y'' = +48x^{-3} \]

The third derivative

\[ y''' = -192x^{-5} \]