Section 4.2: Shortcuts and Rates of Change

NOTATION:

The following are all different ways of writing the derivative of $y = f(x)$:

$$ f'(x), \frac{dy}{dx}, \frac{d[f(x)]}{dx}, y' $$

Here are some short cuts for finding derivatives. These are provided without proof.

**Derivative of a constant:** If $c$ is a constant then

$$ \frac{d}{dx} [c] = 0. $$

**Ex 1:**

a) $y = 7$ then $\frac{dy}{dx} = 0$

b) $y = \pi$ then $\frac{dy}{dx} = 0$

**The power rule:** If $n$ is any real number then

$$ \frac{d}{dx} [x^n] = nx^{n-1} $$

**Ex 2:**

a) $y = x^3$ then $\frac{dy}{dx} = 3x^{3-1} = 3x^2$

b) $y = \sqrt{x^3}$

For this problem we need to rewrite the exponent so it is of the correct form:

$y = x^{3/2}$ and then the derivative is:

$$ \frac{dy}{dx} = \frac{d[x^{3/2}]}{dx} = \frac{3}{2}x^{1/2} $$

**The constant multiple rule:** If $c$ is a constant and $f(x)$ is a differentiable function then

$$ \frac{d}{dx} [cf(x)] = c \frac{d}{dx} [f(x)] $$
Ex 3:  

a) \( y = 7x^3 \) then \( \frac{dy}{dx} = 7(3x^{3-1}) = 21x^2 \)

b) \( y = \frac{4}{x^3} \)

For this problem we need to rewrite the exponent so it is of the correct form:

\( y = 4x^{-3} \) Then the derivative is:

\[
\frac{dy}{dx} = d[4x^{-3}] = 4(-3x^{-3-1}) = -12x^{-4} = \frac{-12}{x^4}
\]

The sum and difference rule: If \( f(x) \) and \( g(x) \) are differentiable functions then

\[
\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)]
\]

Ex 4:  

a) \( y = 7x^3 + \frac{4}{(2x)^3} \)

For this problem we need to rewrite the exponent so it is of the correct form:

\( y = 7x^3 + \frac{4}{8x^3} = 7x^3 + \frac{1}{2}x^{-3} \)

then

\[
\frac{dy}{dx} = 7(3x^{3-1}) + (-3)\left(\frac{1}{2}x^{-4}\right)
\]

Simplify

\[
\frac{dy}{dx} = 21x^2 - \left(\frac{3}{2}x^{-4}\right)
\]

b) \( y = \frac{4}{3x^3} + 4\sqrt{x^5} \)

Rewrite

\( y = \frac{4}{3}x^{-3} + 4x^{5/2} \)

then

\[
y' = \frac{4}{3}(-3x^{-3-1}) + 4\left(\frac{5}{2}x^{3/2}\right)
\]

\[
y' = \frac{-12}{3}x^{-4} + 10x^{3/2}
\]
Ex 5: Find the tangent line at the point:

a. \( f(x) = -2, \quad (2, -2) \)

b. \( f(x) = \sqrt[4]{x}, \quad (16, 2) \)

c. \( f(x) = 8 - x^3, \quad (2, 0) \)

d. \( f(x) = \frac{2}{(3x)^2}, \quad (1, \frac{2}{9}) \)
Ex 6: Find the derivative:

a. \( f(x) = x^2 - 3x - 3x^{-2} \)

b. \( f(x) = t^{2/3} - t^{1/3} + 4 \)

c. \( f(x) = 3x(6x - 5x^2) \)

d. \( f(x) = \frac{2x^2 - 3x + 1}{x} \)  
   Simplify: \( f(x) = 2x - 3 + \frac{1}{x} = 2x - 3 + x^{-1} \)

Ex 7: Find where the function has a horizontal tangent line:
\( y = x^3 + x \)
Rates of Change:

Average rate of change: Slope of the secant line:

\[
\text{Avg. rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Instantaneous rate of change: Slope of the tangent line (DERIVATIVE)

Ex 8.
a) Find the average rate of change on \([2, 2.1]\) for the function \(f(t) = t^2 - 3\).

Avg. rate of change = \[
\frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1
\]

b) Find the instantaneous rate of change at \(t = 2\).

Use the derivative: \(f'(2)\)

\[
f'(t) = 2t
\]

\[
f'(2) = 4
\]

Projectile Motion

On the earth, neglecting air resistance the position of an object moving straight up and down is given by the following equations: (One in standard and one in metric)

Here \(v_0 = \) initial velocity (or velocity at time 0) and \(s_0 = \) initial position (or position at time 0)

\[
s(t) = -16t^2 + v_0 t + s_0 \quad \text{Standard units of measure}
\]
\[
s(t) = -4.9t^2 + v_0 t + s_0 \quad \text{Metric units of measure}
\]

The **Velocity** is the derivative of the position: \(v(t) = s'(t)\)
Ex 9: A ball is thrown straight down from the top of a 220 foot building with an initial velocity of -22 feet per second. What is the velocity after 3 seconds? What is the velocity after falling 108 feet.