Section 4.1: The Derivative and Tangent Line Problem

Recall: For the slope of a line we need two points \((x_1, y_1)\) and \((x_2, y_2)\). Then the slope is given by the formula:

\[
m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

On a curve we can find the slope of a Secant Line because a secant passes through the curve at least two times. In the graph below the red line is a secant line. The points of intersection are \((x_1, f(x_1))\) and \((x_2, f(x_2))\).

So the slope can be written: \[
m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Or if we let \(x_2 = x_1 + \Delta x\) then the equation can be written:

\[
m = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}
\]

What about the slope of the tangent line at \(x_i\)?
Now we don’t have two points to find the slope we only know \((x_1, f(x_1))\).

So what can we do? We can find another point on the curve that is CLOSE to \((x_1, f(x_1))\) and use that to find the slope.

How CLOSE? \(\lim_{\Delta x \to 0} \) CLOSE.

To find the slope of the tangent line we take the limit of the slope of the secant line:

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}
\]

If we do this in general for a generic \(x\) then we get the DERIVATIVE of the function. The Derivative is a function (denoted \(f'(x)\)) which will tell you the SLOPE OF THE TANGENT LINE at any point on the function. It is given by the following formula:

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]
Ex 1: Find the slope of the tangent line to \( f(x) = 5 - x^2 \) at the point (2, 1)

In this case \((x_1, f(x_1)) = (2, 1)\)

SLOPE of the tangent line \(= \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \)

\(= \lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} \)

\(= \lim_{\Delta x \to 0} \frac{5 - (2 + \Delta x)^2 - [5 - 2^2]}{\Delta x} \)

\[ = -4 \]

EX 2: Find the derivative and the tangent line at the point.

a) \( f(x) = x^3 + x^2 \) at the point (1, 2)

b) \( f(x) = 2x^2 + x - 1 \) at the point (-1, 0)
c) \( f(x) = \frac{4}{\sqrt{x}} \) at the point (4, 2)

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{4}{\sqrt{x + \Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x}
\]

Clear the denominator:

\[
= \lim_{\Delta x \to 0} \frac{4(\sqrt{x}) - 4(\sqrt{x + \Delta x})}{\Delta x(\sqrt{x + \Delta x})(\sqrt{x})}
\]

Rationalize the numerator:

\[
= \lim_{\Delta x \to 0} \frac{4(\sqrt{x}) - 4(\sqrt{x + \Delta x})(\sqrt{x + \Delta x} + \sqrt{x})}{\Delta x(\sqrt{x + \Delta x})(\sqrt{x})}
\]

\[
= \lim_{\Delta x \to 0} \frac{4x - 4(x + \Delta x)}{\Delta x(\sqrt{x + \Delta x})(\sqrt{x + \Delta x} + \sqrt{x})}
\]

Simplify:

\[
= \lim_{\Delta x \to 0} \frac{-4}{\Delta x(\sqrt{x + \Delta x})(\sqrt{x} + \sqrt{x + \Delta x})}
\]

Let \( \Delta x = 0 \):

\[
f'(x) = \frac{-4}{2x\sqrt{x}}
\]

Now we need to find the tangent line so we need the slope and a point.

The point is (4, 2) and the slope comes from the derivative \( m = f'(4) = -1/4 \).

The equation of the line is

\[
y - y_1 = m(x - x_1)
\]

\[
y - 2 = -\frac{1}{4}(x - 4)
\]

\[
y = -\frac{1}{4}x + 3
\]
Ex 3 Find the equation of a line that is tangent to the graph of \( f(x) = x^3 + 2 \) and parallel to the line \( 3x - y - 4 = 0 \)

Need a tangent line so we need two things:

a) point  
b) slope

Find the slope from the line \( y = 3x - 4 \): the SLOPE is \( m = 3 \)

Need the point on \( f(x) \) that has slope 3 so we need to solve for the derivative (SLOPE of the Tangent line) and set it equal to 3

Solve:

\[
3 = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2 - (x^3 + 2)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 2 - x^3 - 2}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \Delta x \left( 3x^2 + 3x(\Delta x) + (\Delta x)^2 \right)
\]

\[
= 3x^2
\]

So we solve the equation

\[
3 = 3x^2
\]

\[
x = \pm 1
\]

\[
x = +1: \ f(1) = 2 \quad \Rightarrow \ y - 2 = 3(x - 1)
\]

\[
x = -1: \ f(-1) = 1 \quad \Rightarrow \ y - 1 = 3(x + 1)
\]