Section 10.1 Using Fundamental Identities

Recall these identities:

\[ \tan(\theta) = \frac{\sin \theta}{\cos \theta} \quad \text{csc}(\theta) = \frac{1}{\sin \theta} \]
\[ \sec(\theta) = \frac{1}{\cos \theta} \quad \cot(\theta) = \frac{\cos \theta}{\sin \theta} \]

Pythagorean identities:

\[ \sin^2(\theta) + \cos^2(\theta) = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta \]

We can find all of the trigonometric functions of an angle without drawing a triangle if we are given sufficient information and we know the trigonometric identities. By sufficient information we mean two trigonometric functions of the angle which are not reciprocals of each other.

**Ex 1:** \( \tan x = \frac{\sqrt{3}}{3} \) and \( \cos x = -\frac{\sqrt{3}}{2} \) find the other trigonometric functions.

Since all of our trigonometric functions can be expressed in terms of sine and cosine it is easiest to find both of those functions first.

\[ \tan x = \frac{\sqrt{3}}{3} = \frac{\sin x}{\cos x} = \frac{-\sqrt{3}}{2} \]

We now have an equation to solve for sine:

\[ \frac{\sqrt{3}}{3} = \frac{\sin x}{\sqrt{3}} \Rightarrow \sin x = \frac{\sqrt{3}}{3} \left( -\frac{\sqrt{3}}{2} \right) = -\frac{1}{2} \]

From here we can solve for the other three by taking reciprocals:

\[ \csc x = -2 \]
\[ \sec x = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3} \]
\[ \cot x = \frac{3}{\sqrt{3}} = \sqrt{3} \]
We can also use the identities to take complicated expressions and make them simpler. This is known as simplification.

**Ex 2:** Simplify $\cot^2 x - \csc^2 x$

In this example we have many terms squared so it is going to be easiest to try to use one of the Pythagorean identities. In this case we will use $1 + \cot^2 \theta = \csc^2 \theta$

$$\cot^2 x - \csc^2 x = \cot^2 x - (1 + \cot^2 x) = \cot^2 x - 1 - \cot^2 x = -1$$

So $\cot^2 x - \csc^2 x = -1$

**Ex 3:** Simplify $\frac{\sec^2 x - 1}{\sin^2 x}$

Once again we have squared terms so we need to use a Pythagorean identity: $\tan^2 \theta = \sec^2 \theta - 1$

$$\frac{\sec^2 x - 1}{\sin^2 x} = \frac{\tan^2 x}{\sin^2 x} = \frac{\sin^2 x}{\cos^2 x \sin^2 x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

So $\frac{\sec^2 x - 1}{\sin^2 x} = \sec^2 x$

**Ex 4:** Simplify $\frac{1 - \sin^2 x}{\csc^2 x - 1}$
Sometimes a problem requires factorization as well:

**Ex 5:** Factor and simplify \( \tan^4 x + 2 \tan^2 x + 1 \)

The trick to simplifying this problem to see that it is a quadratic equation in \( \tan^2 x \). To see this more clearly we will do a ”\( u\)-substitution”. In this case we will let \( u = \tan^2 x \). Then we get the following:

\[
(tan^2 x)^2 + 2(tan^2 x) + 1
\]

\[
u^2 + 2u + 1 = (u + 1)^2
\]

but we don’t want a solution in \( u \) so we have to substitute for \( u = \tan^2 x \) to get

\[
(tan^2 x + 1)^2 = (sec^2 x)^2 = sec^4 x
\]

**Ex 6:** Factor and simplify \( \sin^2 x \sec^2 x - \sin^2 x \)

Here we will factor the common factor \( \sin^2 x \) and then apply the identity \( \tan^2 \theta = \sec^2 \theta - 1 \)

\[
\sin^2 x \sec^2 x - \sin^2 x = \sin^2 x(\sec^2 x - 1) = \sin^2 x \tan^2 x
\]

**Ex 7:** Simplify \( \frac{1}{\sec x + 1} = \frac{1}{\sec x - 1} \)

Here we need to find a common denominator.