INTRO: In this Module we will introduce four different kinds of problems. First we will discuss projectile problems when one of the initial conditions is not given (for example, the initial speed of the projectile, or the angle it is fired). These we call missing parameter problems and are not very different from high school algebra problems, where an unknown quantity was given a name such as $x$ and solved for at the end of the problem. Then we will discuss variable acceleration problems. These will be posed as non-earthly projectile problems, since we already understand the machinery to solve projectile problems, but the real reason for interest in such problems is as an introduction to problems with non-constant forces. Next, we will learn how to measure the distance along a curve, i.e., arc length. Finally, we will begin the study of coordinate systems aligned with trajectories.

0. MISSING PARAMETER PROBLEMS

If all initial conditions are not given, so that the constants $c_1, c_2$ and $p_1, p_2$ can not all be determined, leave those which can not be determined as parameters and find them later (usually at the very end of the solution) from other data. Problems like this are called missing parameter problems.

Example: A projectile is fired at a 30 degree angle from the ground, its initial speed is not given, but the range is 500 feet. What was its initial speed?

We will proceed to carry out the three standard steps. However, since we do not know the initial speed, we can not exactly write down all the I.C.’s. To surmount this, we will give a name to the initial speed. We call it $v$. Now we proceed with the three standard steps.
FIRST the Initial Conditions:
\( \vec{v}(0) = \langle v \cos 30^\circ, v \sin 30^\circ \rangle = \langle \frac{\sqrt{3}}{2}v, \frac{1}{2}v \rangle \)
\( \vec{r}(0) = \langle 0, 0 \rangle \)

NEXT the Velocity and Position
\( \vec{a} = \langle 0, -32 \rangle \)
\( \vec{v}(t) = \langle c_1, -32t + c_2 \rangle = \langle \frac{\sqrt{3}}{2}v, -32t + \frac{1}{2}v \rangle \)
\( \vec{r}(t) = \langle \frac{\sqrt{3}}{2}vt + p_1, -16t^2 + \frac{1}{2}vt + p_2 \rangle = \langle \frac{\sqrt{3}}{2}vt, -16t^2 + \frac{1}{2}vt \rangle \)

FINALLY to find \( t \) for the range:
\( y(t) = 0 \implies -16t^2 + \frac{1}{2}vt = 0 \implies t = \frac{v}{32} \)

Then, insert \( t \) in \( x(t) \) so you can set this equal to 500 feet. The only variable you will not know is \( v \).
\( x(t) = \frac{\sqrt{3}}{2}v \cdot \frac{v}{32} = v^2 \frac{\sqrt{3}}{64} = 500 \implies v = \sqrt{\frac{32000}{\sqrt{3}}} \)

1. VARIABLE ACCELERATION

This is best explained with an example. As you will see, the only difference between these problems and traditional trajectory problems is that in these problems the acceleration will not be a constant \([< 0, -9.8 > \) or \(< 0, -32 >, \) depending upon units], but rather a given function of \( t \). The reason we study these is not to torment you with fictional problems about mythical planets, but rather because, according to Newton’s law, if a force on a given mass is time-dependent, then the acceleration will be time dependent. Indeed, \( \vec{a} = \frac{1}{m} \vec{F}, \) rather than \( \vec{F} = m\vec{a}, \) is usually how Newton’s famous law is implemented in physics problems.

**Example:** The acceleration of a particle in an experiment is given as \( \vec{a}(t) = \langle 0, -t^2 \rangle. \) Find the horizontal distance traveled by the particle until it hits the ground again if it starts from the ground at an angle of 45 degrees with an initial speed of 10 ft/sec.

FIRST: Initial Conditions
\( \vec{r}(0) = \langle 0, 0 \rangle \)
\( \vec{v}(0) = \langle 10 \frac{\sqrt{2}}{2}, 10 \frac{\sqrt{2}}{2} \rangle = \langle 5\sqrt{2}, 5\sqrt{2} \rangle \)
Next: Position and Velocity

\[ \vec{v}(t) = < c_1, -\frac{1}{3}t^3 + c_2 > = < 5\sqrt{2}, -\frac{1}{3}t^3 + 5\sqrt{2} > \]

\[ \vec{r}(t) = < 5\sqrt{2}t + p_1, -\frac{1}{12}t^4 + 5\sqrt{2}t + p_2 > = < 5\sqrt{2}t, -\frac{1}{12}t^4 + 5\sqrt{2}t > \]

Finally: To find the range

\[ y(t) = 0 \implies t = (60\sqrt{2})^{\frac{1}{3}} \implies x(t) = 5\sqrt{2} \cdot (60\sqrt{2})^{\frac{1}{3}} = \text{range} \]

2. ARC LENGTH

Arc length along trajectory \( \vec{r}(t) \) from \( t_1 \) to \( t_2 \) is:

\[
\text{arclength} = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

If it helps any, you can remember this as

\[
\text{arclength} = \int_{t_1}^{t_2} ||\vec{v}(t)|| \, dt
\]

This is a formula you must know. (Obviously, in two dimensions there will be no \( z'(t) \).)

Complications: There are two types of complications to implementing this formula.

First type:

It is easiest if you are given the trajectory already parametrized, as \( \vec{r}(t) = < x(t), y(t) > \). However, if you are given the trajectory in Cartesian form, e.g., \( y = 2x^2 + 3x \), then you will have to parametrize it yourself, in this case \( \vec{r}(t) = < t, 2t^2 + 3t > \).

Second type:

It is easiest if the arclength is requested between two values of the parameter \( t \), say from \( t_1 \) to \( t_2 \) as in the formula. However, it is more common that the arclength is requested between two points of the curve given in Cartesian coordinates. Then you will have to find the values of \( t \) corresponding to these points.

Example 1: Find the length of the trajectory \( \vec{r}(t) = < \frac{4t}{\pi}, \cos 2t, \sin 2t > \) from the point \( < 1, 0, 1 > \) to the point \( < 4, 1, 0 > \).

It is rather easy to see that these points correspond to \( t = \frac{\pi}{4} \) and \( t = \pi \). Then the arclength is:

\[
\int_{\pi/4}^{\pi} \sqrt{(4/\pi)^2 + 4 \sin^2 2t + 4 \cos^2 2t} \, dt = \int_{\pi/4}^{\pi} \sqrt{(4/\pi)^2 + 4} \, dt = \frac{3\pi}{4} \sqrt{(4/\pi)^2 + 4}
\]
Example 2: Find coordinates of point a distance 0.5 (as measured along the curve) from point (2,0) in positive $t$ direction on curve \[
\begin{align*}
x &= 2 \sin t \\
y &= 2 \cos t
\end{align*}.
\]
Solution: Point (2,0) corresponds to $t_1 = \frac{\pi}{2}$. Point we are looking for we will let correspond to $t_2$. (This is a little like the missing parameter trajectory problems. Since we are not given the second point, we make up a name $t_2$ for its location, and then proceed with the problem.)

Distance from $t_1$ to $t_2$ is:
\[
\int_{\frac{\pi}{2}}^{t_2} \sqrt{4 \cos^2 t + 4 \sin^2 t} \, dt = \int_{\frac{\pi}{2}}^{t_2} 2 \, dt = 2(t_2 - \frac{\pi}{2}) = 0.5
\]
\[
\implies t_2 = \frac{\pi}{2} + \frac{1}{4}
\]
So, answer: \[
\begin{align*}
x &= 2 \sin(\frac{\pi}{2} + \frac{1}{4}) \\
y &= 2 \cos(\frac{\pi}{2} + \frac{1}{4})
\end{align*}
\]
To reiterate: in this problem we knew the initial value for $t$, but we did not know the final value – in fact, we were trying to locate it in order to answer the question. Therefore, like any good student of high school algebra we gave this final value of $t$ a name, $t_2$, and it was easy to write an equality involving $t_2$. That eventually led to a value for $t_2$.

Example 3: Coordinates of point on the curve $y = x^2$ a distance 1 from the origin (as measured along the curve) and with $x$ positive.

Solution: First parametrize the curve: $\vec{r}(t) = <t, t^2>$ The origin corresponds to $t = 0$. Let $t_2$ give the location of the point being sought. Then we would need to solve:
\[
\int_0^{t_2} \sqrt{1 + 4t^2} \, dt = 1
\]

3. ORTHOGONAL UNIT VECTORS ON TRAJECTORIES

UNIT TANGENT VECTOR

Vector tangent to trajectory $\vec{r}(t)$ of unit length is:
\[
\hat{T}(t) = \frac{1}{||\vec{v}(t)||} \vec{v}(t) = \frac{1}{||\vec{r}'(t)||} \vec{r}'(t) \quad \text{(unit tangent vector)}
\]
Example 1: \( \vec{r}(t) = <t^2, 1, e^t> \)
\( \vec{v}(t) = <2t, 0, e^t> \)
\( \hat{T}(t) = \frac{1}{\sqrt{4t^2 + e^{2t}}} <2t, 0, e^t> \)

Example 2: Unit tangent vector to \( \vec{r}(t) = <t^2, t, 2t> \) at \( t = 1 \).
Answer: \( \hat{T} = <\frac{2}{3}, 1, \frac{2}{3}> \)

UNIT NORMAL VECTOR
\[
\hat{N} = \frac{1}{||T'(t)||} T'(t) \quad \text{(unit normal vector)}
\]

\( \hat{T} \cdot \hat{N} = 0 \) so they are perpendicular.

To see this, note \( \hat{T} \cdot \hat{T} = 1 \). Since derivative of a constant is 0, \( \frac{d}{dt} \hat{T} \cdot \hat{T} = 0 \) Since product rule works here, \( \frac{d}{dt} \hat{T} \cdot \hat{T} = \hat{T}' \cdot \hat{T} + \hat{T} \cdot \hat{T}' = 2\hat{T} \cdot \hat{T}' \). But \( \hat{T}' \) and \( \hat{N} \) point in the same direction, so if \( \hat{T} \cdot \hat{T}' = 0 \), so does \( \hat{T} \cdot \hat{N} \).

\( \hat{T} \) points tangent to the trajectory in the direction of motion.
\( \hat{N} \) points toward the center of ”instantaneous circle” which trajectory is making.

The strategy for computing \( \hat{N} \) is clear: differentiate the trajectory \( \vec{r}(t) \) to get the velocity \( \vec{v}(t) = \vec{r}'(t) \), normalize it (ie, make it of length 1) by multiplying by \( \frac{1}{||\vec{v}(t)||} \), thus obtaining the tangent vector \( \hat{T}(t) = \frac{1}{||\vec{v}(t)||} \vec{v}(t) \), differentiate the tangent vector to obtain \( \hat{T}'(t) \), which is guaranteed to be pointing in a direction perpendicular to \( \hat{T} \), and finally normalizing again: \( \hat{N}(t) = \frac{1}{||\hat{T}'(t)||} \hat{T}'(t) \) to make vector unit length. (You should study this sentence until you understand what it is saying)

Unfortunately, the calculation can get terribly messy, primarily because the product rule usually has to be used to differentiate \( \hat{T} \), which can lead to so many terms that it is difficult to see what you have obtained after the subsequent normalization of \( \hat{T}'(t) \). We will understand \( \hat{N} \) thoroughly, but will not often have to actually compute it.

Obviously, a third vector perpendicular to both \( \hat{T} \) and \( \hat{N} \) is:
\[
\hat{B} = \hat{T} \times \hat{N} \quad \text{(unit binormal vector)}
\]
(In two dimensions, one has only \( \hat{T} \) and \( \hat{N} \).) \( \hat{T}, \hat{N}, \hat{B} \) are a complete set of perpendicular vectors moving along the trajectory – a moving coordinate system – important in a number
of applications. We will deal only with one application: breaking up acceleration $\vec{a}(t)$ into its tangential component $a_T$ and its normal component $a_N$:

$$\vec{a}(t) = a_T(t) \hat{T} + a_N(t) \hat{N}$$

but we will not do this until the next lecture.

QUIZ PREP

1. Standard projectile problems (again)

   **Warning:** always check if units are feet or meters.

   **Warning:** angles may be given in degrees or radians. Be careful with your calculator (Sliderule) when you evaluate a trig function that the calculator is correctly set for degrees or radians, as necessary.

2. Missing parameter projectile problems

3. Variable acceleration problems

   **Warning:** acceleration of magnitude $m$ in *downward direction* means $\vec{a} = <0, -m>$

4. Length of a curve problems

5. Unit tangent vectors