Technical Mathematics

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Chapter 1

The Number System and Algebraic Operations

1.1 Numbers

A fundamental aspect to our work in math is an understanding of the different types of numbers. One type of numbers you use every day is the **counting numbers** , or **natural numbers**. These numbers are $\{1, 2, 3, 4, \cdots\}$. Another set of numbers you use everyday to measure, such as while cooking or building somthing, are called **rational numbers** (i.e. $\frac{1}{2}$ cup, or $\frac{3}{4}$ inch).Since we're going to be talking about the different *sets* of numbers in mathematics we might as well begin with using the correct notation when referring to these sets. In mathematics, there are several ways of defining a set, either by listing the numbers or members in the set, using set-builder notation, or interval notation. Listing the **objects** or **members** of the set can be seen example 1.1.1, while another could be to verbally describe the set which is typically done using **set-builder** notation as can be seen in example 1.1.2. The last method that we'll be looking at is **interval notation** where the list of numbers is bound between two other numbers as in example 1.1.3. In any case, a set is always defined by using *curly brackets* or just *brackets*, { }, with the exception of interval notation.

Example 1.1.1:

Listing Method

$\{a,\ e,\ i,\ o,\ u\}$

$\{1, 1, 2, 3, 5, 8, 13, 21\}$

$\{2, 5, 3, 8, 14, 1, 7\}$

notice that when listing the elements in a set that the order doesn't matter. In this book most of the sets that we deal with will be specific sets of numbers such as integers, or rational numbers, etc.

1.1.1 Integers

The **whole numbers** is the set of natural numbers that includes 0. However, some problems require the following arithmetic $2 - 6 = -4$ where -4 is not in the set of whole numbers, thus requiring the use of a new set of numbers called **integers** that includes negative values. The following set is the set of integers defined using the listing method.

$$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

The negative numbers are called the **negative integers** and the natural numbers are called the **postive integers**. The number 0 is neither positive nor negative. Positive numbers can be written with a plus sign in front of them as $+1, +2, +3, \cdots$ but this is not necessary nor typically practiced. The numbers $\{0, 1, 2, 3, \cdots\}$ are called **nonnegative integers** or **whole numbers**.

Example 1.1.2:

Set-Builder

$\{x \mid x \text{ is a real number}\}$

$\{x \mid x \text{ is an integer}\}$

The way to read or describe a set in set-builder notation is to begin on the left. Looking at the first set in the examples says: all values of x such that x is a real number.

1.1.2 Rational Numbers

Rational numbers are used to represent the division, or ratio, of one integer by a another integer. There are both positive and negative rational numbers. Rational numbers can be used to express nearly any number in more than one way. For example, the numbers $\frac{4}{2}, \frac{8}{4}, \frac{2}{1}$, and $\frac{242}{121}$ are all different ways of representing the number 2. In addition, any terminating, or repeating decimal is a rational number since it can be written as a ratio of integers (i.e. $0.3\overline{3} = \frac{1}{3}$) . The following is a list of some rational numbers including 0 since the division of zero and any other number, including integers, will always simplify to zero.

$$\left\{0,\ 3,\ \frac{22}{5},\ -\frac{7}{3},\ \frac{289}{17},\ \frac{-1}{3},\ \frac{28}{14},\ 0.3,\ 1.23,\ 0.3\overline{3}\right\}$$

Example 1.1.3:

Interval Notation

$(-\infty, \infty)$ = all real numbers

$[0, 10)$ = all numbers beginning at 0 and up to 10 but not including 10.

In interval notation we use [or] to include a number, and a (or) to exclude a number.

1.1.3 Irrational Numbers

Irrational numbers are numbers that can *not* be represented as the ratio of two integers. It was once thought that *any* number could be written as a rational number and it was eventually proved that not all numbers (i.e. $\sqrt{2}$) can not be written as the division of two integers, either positive or negative. In complete contrast to the early belief, it has since been proven that there are just as many irrational numbers as there are rational ones. The following are examples of just a few irrational numbers.

Example 1.1.4:

$$3.141592653589793238462643...$$

π is an irrational number with non-terminating and non-repeating decimal values.

$$\left\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi, -\sqrt{17}\right\}$$

An irrational number is best described as a number whose decimal representation is both non-repeating and non-terminating as illustrated in example 1.1.4.

1.1.4 Real numbers

When rational numbers are combined with irrational numbers, we get the set of **real numbers**. For the most part the work done in this course will involve the set of real numbers; however, we will be covering the *complex* numbering system as well which involve imaginary numbers.

At times it is convenient to represent the real numbers on a line called the **number line** as illustrated below. The typical number line is a horizontal line that has been marked in equally spaced intervals. One of these marks is called the **origin** and is indicated by the number zero (0). The marks to the right are labeled with positive integers and the marks to the left of the origin are marked with negative integers.

Even though all the other numbers are not visibly present on the number line such as π, $\sqrt{2}$, φ, or any other irrational number, those numbers are implied. Of course we could always indicate approximately where those numbers are on the number line.

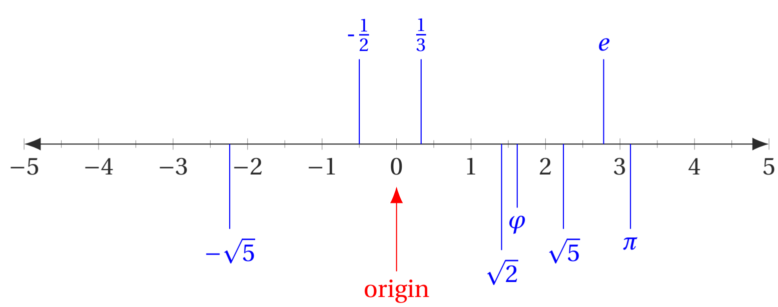


Figure 1.1: Number Line

Symbols for Sets of Numbers

Throughout this book, and mathematics in general, we will often reference specific sets of numbers, such as integers or rationals, often. However, the listing method shown above to reference such sets gets cumbersome and tedious after a while so from now on I will refer to the following universal symbols to represent each of the following sets:

Symbol	Name	Set/Description
$\mathbb{N}$	Naturals	$\{1,2,3,\cdots\}$
$\mathbb{Z}$	Integers	$\{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$
$\mathbb{Q}$	Rationals	$\left\{\frac{p}{q}\,\middle\|\, p, \text{and } q \text{ are integers and } q \neq 0\right\}$
$\mathbb{R}$	Reals	$\left\{x \,\middle\|\, \text{x can be either rational or irrational}\right\}$
$\mathbb{R}\backslash\mathbb{Q}$	Irrationals	$\left\{x \,\middle\|\, \text{x can not be represented as the ratio of two integers}\right\}$

note: There is no symbol designated for the set of irrational numbers so we take the set of real numbers and subtract the rational numbers which leaves the irrational numbers as represented by $\mathbb{R}\backslash\mathbb{Q}$. The backslash (\) is traditionally used for subtraction in set arithmetic although the more familiar symbol minus (-) has also been used.

Example 1.1.5:

To show that a value is among a set, or a member of a set, the $\in$ symbol is used as in the following examples:

$3 \in \mathbb{Z}$ says that 3 is in the set of integers, or 3 is an integer.

$\sqrt{3} \in \mathbb{R}\backslash\mathbb{Q}$ says that $\sqrt{3}$ is an irrational number.

$n \in \mathbb{R}$ says that n is a real number, or n is in the set of real numbers.

1.1.5 Decimals

Each real number can be represented by a decimal number. The rational numbers are represented by repeating decimals, and irrational numbers are represented by non-repeating decimals. The decimal representation of a real number allows us to position the number accurately on the number line.

Repeating decimals are decimals that have a value that is repeated indefinitely as in Table 1.1.

Sometimes repeating decimals are represented with a bar over top of the part that is repeated instead of writing out enough decimal places to *imply* which number(s) is/are repeated. If *accuracy* and *precision* are important, which they usually are, then simply writing out enough decimal places to suggest the number that's repeated is unacceptable since this will throw off both the accuracy and precision of the number (more on this in section 1.3). To overcome this, repeating decimals are typically written with a *bar* over top the digit(s) that is/are are repeating. For example, if we use the repeating decimals in Table 1.1 the values would look like $\frac{1}{2} = 0.5\overline{0}$, $\frac{17}{4} = 4.25\overline{0}$, $\frac{1}{3} = 0.\overline{33}$, $\frac{41}{11} = 3.\overline{72}$.

Number	Decimal	Repeats
$\frac{1}{2}$	$= 0.5000\cdots$	0
$\frac{13}{4}$	$= 3.2500\cdots$	0
$\frac{1}{3}$	$= 0.3333\cdots$	3
$\frac{41}{11}$	$= 3.727272\cdots$	72

Table 1.1: Repeated Decimals

Numbers that have zero as the repeating number are called **terminating decimals** and are typically written *without* indicating the repeating zero unless its necessary to indicate precision. For example, we write $\frac{1}{2} = 0.5$ and $\frac{17}{4} = 4.25$. Terminating decimals are a special type of decimal and are important because they can be given an *exact* decimal representation where non-terminating decimals can only have an *approximate* decimal representation which are called irrational numbers.

Irrational numbers are represented by non-repeating and nonterminating decimals. For example, below are the decimal representations of four irrational numbers

$$\sqrt{2} = 1.414213... \qquad \sqrt{7} = 2.6457513...$$

$$\pi = 3.141592653589... \qquad -\frac{15}{2} = -1.93649167...$$

When adding an irrational number to a rational number you will always end up with an irrational number as the result. However, this is not to imply that the addition of any number with an irrational number will result in an irrational number. For example, if you take $\sqrt{3} - \sqrt{3}$ we get a result of zero which is a rational number. The same goes for the product of certain rational numbers. Take $\sqrt{3} * \sqrt{3}$ and we get $\sqrt{3}^2 = 3$ as a result which is rational, so do not make the mistake of thinking that any time we perform arithmetic on an irrational number that the result must also be irrational.

1.1.6 Fractions

The numbers $\dfrac{3}{2}$ and $\dfrac{-1}{1000}$ are both fractions. On the number line these numbers are what appear in between the integers.

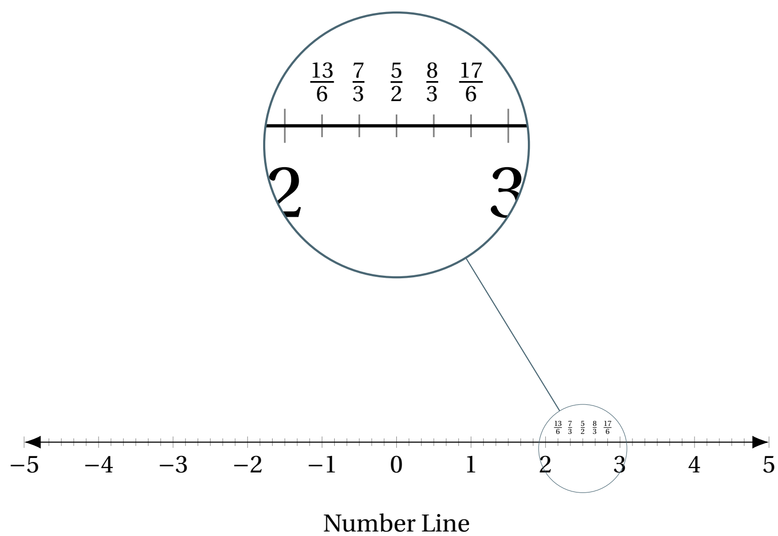


Figure 1.2: Numberline Depicting Fractions

In the picture above, a *fraction* is described as a number that appears in between two integers on a number line. More specifically a **fraction** is part of a whole, but a whole what? The whole can be represented as just about anything that can be divided up into parts such as a circle (pie), or a length of something such as a ruler. The bottom number of a fraction is called the *denominator* and the top number is called the *numerator*. The **denominator** (bottom) indicates how many parts the whole is divided up, while the **numerator** (top) indicates the position the fraction is with respect to the denominator. In figure 1.3 is a ruler showing 16^{th} in marks which means that between each whole number such as 0 and 1, or 1 and 2, that the distance between those two numbers has been divided into 16 equal partitions. The first segment would be $\frac{1}{16}$ the second segment is located at $\frac{2}{16}$, then $\frac{3}{16}$, and $\frac{4}{16}$ etc. These fractions are reducible such as $\frac{4}{16} = \frac{1}{4}$, and what this means is that if we took the distance from 0 and 1 and divided it into 4 equal partitions, then $\frac{1}{4}$ and $\frac{4}{16}$ would be located at the exact same location on the ruler. It is important to understand the details of a fraction, and the relationship between the numerator and the denominator. From here we can see why its necessary to find a common denominator before we can add or subtract two fractions because the denominator is essentially defining a unit of measurement, and like all units of measurement such as feet (ft) , or kilometers (km), we must make sure that all of our measurements are in the same units before we combine them.

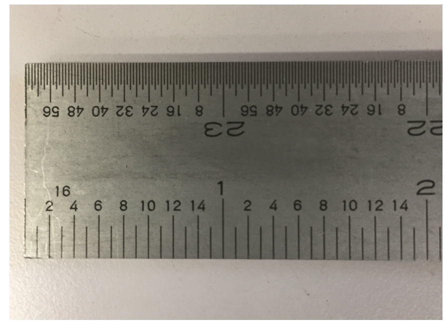


Figure 1.3: Ruler with 16^{th} in. marks

1.1.7 Absolute Value

One of the basic ideas is how far from 0 (the origin) a number is. This idea is called the absolute value, denoted as $|a|$, which will effectively make any number positive to indicate the distance that value is from the origin. In figure 1.4 we can see that the absolute value of a, $|a|$, is the same on either side of the origin. the distance that a or $-a$ is from zero is the absolute value of a, and all distances are positive.

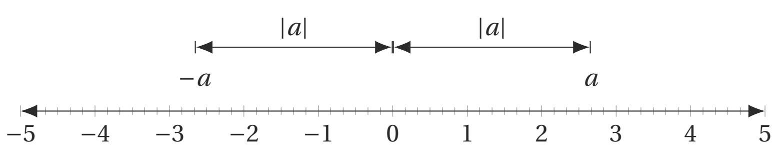


Figure 1.4

Example 1.1.6:

$$|-4| = 4, \qquad |4| = 4$$

$$|-1.618| = 1.618, \qquad |1.618| = 1.618$$

$$|\sqrt{7}| = \sqrt{7}, \qquad |-\sqrt{7}| = \sqrt{7}$$

1.1.8 Properties of Real Numbers

Though it's important to know the different types of sets of numbers, It is just as important if not more so to understand the properties of the real number in order to combine them. These properties are essential in manipulating mathematical expressions and solving equations.

Example 1.1.7:

$$5-3=2, \text{but} \quad 3-5=-2$$
$$-10-2=-12, \text{but} \quad -2-(-10)=8$$
$$4 \div 5 \neq 5 \div 4, \text{or} \quad \frac{4}{5} \neq \frac{5}{4}$$

Properties of Real Numbers

$a+b=b+a$	Commutative property for addition
$ab=ba$	Commutative property for multiplication
$(a+b)+c=a+(b+c)$	Associative property for addition
$(ab)c=a(bc)$	Associative property for multiplication
$a(b+c)=ab+ac$	Distributive property for multiplication over addition
$a+0=a$	Identity element for addition
$a \cdot 1$	Identity element for multiplication

It is important to note that both subtraction and division are not commutative, nor are they associative. As shown in example 1.1.7.

1.1.9 Denominate Numbers

Numbers that are paired with a unit of measurement such as *ft* or *km* are called **denominant numbers**. For example, $10ft$ is called a denominant number because it is has the symbol for feet appended to it. We do not typically spell out *feet* to define a certain number of feet rather we use *ft*, or another common notation is $10'$. Looking closer at the symbol *ft*, or mi, one would assume that these symbols represent abbreviations for both feet and mile respectively; however, they are not abbreviations. Rather, they are symbols, and as a consequence a period does not succeed them. Similarly we would not write out kilometers to represent a denominant number in kilometers; instead we would use *km* such $0.3km$. Table 1.2 lists some common notations used in some different units of measurement, but not a complete list. Most of the units listed here you probably will recognize. As we progress through the various chapters in this book, you will undoubtedly learn more.

Measurment	Unit	Abbreviation
Length	feet	*ft*
Length	inches	*in*
Length	miles	*mi*
Length	kilometers	*km*
Length	meters	*m*
Length	milimeters	*mm*
Mass	miligram	*mg*
Mass	kilograms	*kg*
Mass	grams	*g*
Area	square feet	ft^2
Volume	cubic yards	yd^3

Table 1.2: Units of Measurement

1.2 Fundamental Operations with Real Numbers

In section 1.1 we talked about the properties of real numbers, and the first property that we listed was the commutative property for addition. The commutative property states that if two numbers are added, then it doesn't matter in which order they are added as shown in example 1.2.1. The same goes for the commutative law for multiplication. Any two numbers can be multiplied together regardless of the order which can also be seen in example 1.2.1.

Example 1.2.1:

$$5 + 4 = 4 + 5 = 9$$

$$-2 + (-3) = -3 + (-2) = -5$$

$$3 \times (-2) = -2 \times 3 = -6$$

Keep in mind that all of these properties are very important, especially when attempting to manipulate, or solve equations. One such property is the **distributive law for multiplication** over addition which is illustrated in example 1.2.2.

Example 1.2.2:

$$a\,(b + c) = a \times b + a \times c$$

Notice that there is no times operator between the variable a and the parenthesis , (, in example 1.2.2. In these instances where no operator is present, then multiplication is always implied. Eventually we quit using the times symbol, $\times$, in favor of just a dot, such as $a \cdot b$, to state multiplication mainly due to the fact that we often use the letter x within our equations, and the two symbols look too much alike and may mistake one for the other. It is also common to not use any symbol at all, such $ab = a \times b$, which still means to multiply. Below are a few more examples with real numbers of the distributive law.

1.

$$\begin{aligned} 3\,(4 + 1) &= 3 \times 4 + 3 \times 1 \\ &= 3 \cdot 4 + 3 \cdot 1 \\ &= 12 + 3 \\ &= 15 \end{aligned}$$

same expression with a different symbol for multiplication

2. A good application, among many, for the distributive property is for converting degrees fahrenheit ($^\circ F$) to degrees celsius ($^\circ C$). Using the formula below, lets convert $12^\circ F$ to celsius.

$$T^\circ C = \frac{5}{9}(T^\circ F - 32)$$

Solution:

To solve this problem we only need to replace our given temperature in fahrenheit with the variable in our formula. In this case we replace $T^\circ F$ with 12.

$$\begin{aligned} T^\circ C &= \frac{5}{9}(12-32) \\ &= \frac{5}{9}\cdot 12 - \frac{5}{9}\cdot 32 \\ &= \frac{20}{3} - \frac{160}{9} \\ &= -\frac{100}{9} \\ &\approx -11^\circ C \end{aligned}$$

Note: When using the formula to convert from fahrenheit to celsius, it is assumed that the student aleady knows how to add and multiply fractions. In addition, the $\approx$ symbol is used to denote an approximation. This will occurr anytime we have to round off a decimal value.

1.2.1 Order of Operations

The order of operations is a collection of rules that determine which procedures within a mathematical expression to do first, then second and so forth. For example, multiplication is given higher priority over addition and subtraction. Therefore, we have to perform the multiplication operation before we can add or subtract. In the expression: $1+2\cdot 3$ we would have to perform $2\cdot 3$ first, then add that result to 1. This is illustrated in Example 1.2.3.

Example 1.2.3:

$$\begin{aligned} 1+2\cdot 3 &= 1+6 \\ &= 7 \end{aligned}$$

It's often necessary to perform certain calculations out of order such as adding before multiplying. As in Example 1.2.3 we may have wanted to perform the addition first, and in this case we could have used a grouping symbol such as the parenthesis to accomplish that. Within the *order of operations* we always perform operations within grouping symbols first. In the cases where more than one grouping symbol is used we would perform the inner most group first and work our way outward. For example, if we want to perform the addition first before multiplication in Example 1.2.3, then we would write the expression with parenthesis around $1+2$ such as $(1+2)\cdot 3$ and the result of this would have given us a 9 instead of the previous result of 7.

Order of Operations

1. Operations within a *grouping symbol*, such as parenthesis, numerator and denominator of a fraction, or expressions bound under a radical symbol, $\sqrt{}$, must be performed first.
2. Exponents are calculated second.
3. Multiplication and divisions are performed in the order in which they appear from left to right, and are done before adding and subtracting.
4. Addition and Subtraction are also performed in the order in which they appear from left to right.

There is an acronym that some use to help remember the *order of operations*, PEMDAS (Parentheses, Exponents, Mulitiplication, Division, Addition, Subtraction), but as you become more familiar with the operations most people will be able to perform the operations in the correct order without difficulty.

Example 1.2.4:

$$36 \div (2+4)$$

To evaluate $36 \div (2+4)$ we have to perform the operation within the parenthesis first giving us the expression $36 \div 6$ where we can now divide 36 by 6 which gives $36 \div 6 = 6$

note: Eventually we will stop using the division symbol, ÷, in favor of expressing division in fractional form. Notice that example 1.2.4 can be written as $\frac{36}{2+4}$ where as its not necessary to put parentheses around $2+4$ since the numerator and denominator denote a grouping symbol in itself.

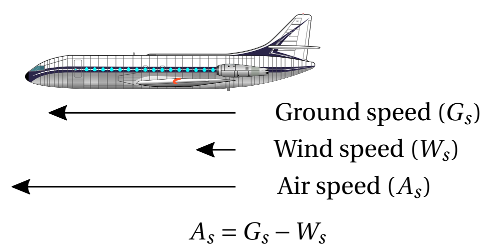
Example 1.2.5:

$$2+8\cdot 2 \div 4$$

In this example we have a little more going on with both multiplication and division; however, the *order of operations* tells us to perform multiplication and division in the order that they appear from left to right, so in this case we have to multiply the 8 and 2 first giving us $2+16\div 4$. Now, we since division comes before addition we calculate $16 \div 4$ next giving us $2+4=6$.

A direct application to adding real numbers is calculating the airspeed (A_s) of a plane. In figure 1.5 is an image of a plane indicating its ground speed (G_s), wind speed (W_s) , and its air speed. The formula for calculating the *airspeed* is given as $A_s = G_s - W_s$, where the airplane is always moving in a positive direction. If the plane is traveling at 600 mph in the same direction as the wind at 30 mph, then the resultant air speed of the plane would be $A_s = 600-30 = 570$ mph.

Figure 1.5: Calculating Airspeed



Example 1.2.6:

Calculate the air speed of a jet traveling with a ground speed of 640 mph with a headwind (opposite direction of plane) of 90 mph.

Solution:

Since the air is travelling in the opposite direction to the plane, then we assign a value that is negative for the wind speed. This is done since we assume that the jet is always travelling in the positive direction, thus the $W_s = -90$ and $G_s = 640$.

$$\begin{aligned} A_s &= 640-(-90) \\ &= 640+90 \\ &= 730 \text{ mph} \end{aligned}$$

note: Anytime we subtract a negative the result is positive.

Example 1.2.7:

Solve $5+36\div 18\times 6-7\times 3$

Solution: Recall that the order of operations require that we perform multiplication and division in the order they appear from left to right, then we can perform the addition and subtraction in the order they appear from left to right.

$$\begin{aligned} 5+36\div 18\times 6-7\times 3 &= 5+2\times 6-7 \\ &= 5+12-7 \\ &= 17-7 \\ &= 10 \end{aligned}$$

Example 1.2.8:

Solve $(14+2(3-6)+5(7-1))$

Solution: We begin with the inner-most parentheses first and work our way out. In this case we have two sets of parentheses that we can evaluate at the same time since one is not nested within the other.

$$\begin{aligned} (14+2(3-6)+5(7-1)) &= (14+2(-3)+5\cdot 6) \\ &= 14-6+30 \\ &= 8+30 \\ &= 38 \end{aligned}$$

1.2.2 Operations with zero

If b is a real number then the following are the properties of zero under the operation of addition, subtraction, multiplication, and division:

$$\begin{aligned} & b+0=b \\ & b-0=b \quad 0-b=-b \\ & b\times 0=0 \\ & 0\div b=0 \text{ which is the same as } \frac{0}{b}=0 \text{ if } b\neq 0 \\ & b\div 0=\frac{b}{0} \Rightarrow \text{ Does not exist} \end{aligned}$$

$b\neq 0$ means that b is not equal to 0. Since it is not possible to divide by 0, we would say that $\frac{a}{0}$ does not exits.

Example 1.2.9:

Evaluate $4 + \frac{0}{2.1} + 5.2 \times 3.7 \times 0$

Solution:

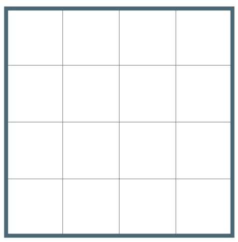
$$\begin{aligned} 4 + \frac{0}{2.1} + 5.2 \times 3.7 \times 0 &= 4 + 0 + 5.2 \times 3.7 \times 0 \\ &= 4 + 19.24 \times 0 \\ &= 4 + 0 \\ &= 4 \end{aligned}$$

1.3 Exponents and Roots

1.3.1 Exponents

It is often necessary to multiply a number by itself either once, or multiple times. For instance, as illustrated in figure 1.6, when we want to determine the area of a square. To calculate the area of a square all we need to do is multiply the length of one of the sides by itself since all sides are the same. In figure 1.6 we can determine the square area in centimeters by multiplying 4 by itself, thus we have 4 cm · 4 cm = 16 square centimeters

Figure 1.6: Square Area



Another way of representing a number being multiplied by itself is by using *exponents*. In the previous example we could denote $4 \cdot 4$ by 4^2. The smaller number 2 written to the upper right of 4 is called the exponent while the larger written number 4 is called the *base*. The base is the number that will be multiplied repeatedly, and the exponent determines how many times the base is multiplied.

Example 1.3.1:

$$\begin{aligned} 2^4 &= 2 \cdot 2 \cdot 2 \cdot 2 && \text{– begin by multiplying from the left} \\ &= 4 \cdot 2 \cdot 2 && \text{– continue multiplying from the left} \\ &= 8 \cdot 2 \\ &= 16 \end{aligned}$$

Here's another example where the base is 5 and the exponent is 3.

Example 1.3.2:

$$\begin{aligned} 5^3 &= 5 \cdot 5 \cdot 5 \\ &= 25 \cdot 5 \\ &= 125 \end{aligned}$$

Of course we can also work the other way. Instead of multiplying a single number multiple times, we can represent that number as an exponent in an attempt to avoid dealing with one large number.

Example 1.3.3:

$$\begin{aligned} 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 &= 3^9 \\ &= 19683 \end{aligned}$$

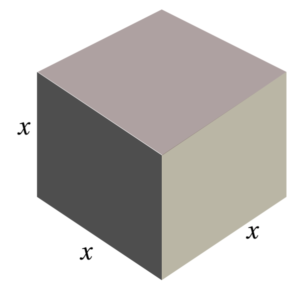
Example 1.3.4:

$$
\begin{aligned}
(-2)^4 &= (-2)(-2)(-2)(-2) \quad && - \text{If no operator is present, then multiplication is implied} \\
&= 4(-2)(-2) \\
&= -8(-2) \\
&= 16
\end{aligned}
$$

The volume, V, of a cube is defined by taking *either* the length, height, or width since they are all the same and multiplying by itself 3 times; which is also known as cubing.

$$V = x^3$$

Figure 1.7: Cube



Example 1.3.5:

How many cubic feet are in one cubic yard of concrete.

Solution: There are 3 feet in one yard, thus we take a base of 3 with an exponent of 3.

$$3^3 = 3 \cdot 3 \cdot 3 = 27$$

Thus, there is 27 cubic feet of concrete in one cubic yard.

Properties of Exponents

1. $a^m \cdot a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}, \quad \text{if } a \neq 0$
3. $(ab)^n = a^n \cdot b^n$
4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad \text{if } b \neq 0$
5. $\left(a^m\right)^n = a^{mn} = \left(a^n\right)^m$
6. $a^0 = 1, \quad \text{if } a \neq 0$
7. $a^{-n} = \dfrac{1}{a^n}$
8. $\sqrt[n]{a^m} = a^{m/n}$

Example 1.3.6:

Use the properties of exponents to verify that $\dfrac{2^6}{2^3} = 2^{6-3} = 2^3 = 8$.

Solution:

$$\frac{2^6}{2^3} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot 2}{\underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}}} = 2^3 = 8$$

Note: In example 1.3.6 we can cancel out the values of 2 because everything in both the numerator and denominator are being multiplied. If there were any values being added or subtracted, then those operations would have to be done first before any cancellation can occur.

Example 1.3.7:

Use property 1. to show that $3^2 \cdot 3^3 = 3^5$.

Solution:

$$3^2 \cdot 3^3 = (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3) = 3^5 = 243$$

Example 1.3.8:

Using properties of exponents, expand $(2x^2y)^3$.

Solution:

$$\begin{aligned}(2x^2y)^3 &= 2^3(x^2)^3y^3 && \text{property 3}\\ &= 8x^{2\cdot 3}y^3 && \text{property 5}\\ &= 8x^6y^3\end{aligned}$$

Example 1.3.9:

Simplify $\dfrac{7^2}{7^{10}}$ with a result that consists of only positive exponents.

Solution:

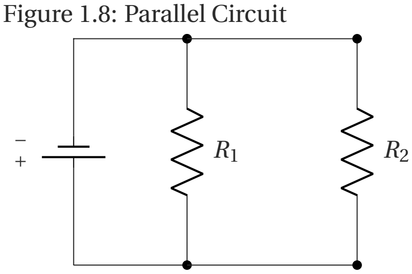
$$\begin{aligned}\frac{7^2}{7^{10}} &= 7^{2-10} && \text{combine the two bases using property 2.}\\ &= 7^{-8}\\ &= \frac{1}{7^8} && \text{use property 7. to rewrite using positive exponents.}\end{aligned}$$

The total resistance of a parallel circuit is given by:

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

An example of a simple parallel circuit is shown in figure 1.8

Figure 1.8: Parallel Circuit



Example 1.3.10:

Simplify $\dfrac{x^3yr^{-2}}{r^{-3}x^5y^{-1}}$ with positive exponents only.

Solution:

$$\begin{aligned}\frac{x^3yr^{-2}}{r^{-3}x^5y^{-1}} &= x^{3-5}y^{1-(-1)}r^{-2-(-3)} && \text{combine like bases using property 2.}\\ &= x^{-2}y^2r && \text{Simplify the exponents. recall } -(-a) = a.\\ &= \frac{y^2r}{x^2} && \text{rewrite as positive exponents using property 7.}\end{aligned}$$

Example 1.3.11:

Find the total resistance, R, of a *parellel circuit* when $R_1 = 300\ \Omega$ and $R_2 = 200\ \Omega$

Solution:

$$
\begin{aligned}
R &= \left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-1} \\
&= \left(\frac{1}{300}+\frac{1}{200}\right)^{-1} && \text{substitute } R_1 = 300 \text{ and } R_2 = 200 \\
&= \left(\frac{200+300}{300\cdot 200}\right)^{-1} && \text{add the two fraction within parenthesis first.} \\
&= \left(\frac{300\cdot 200}{200+300}\right) && \text{the reciprocal of } \frac{a}{b} \text{ is the same as } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \\
&= 120
\end{aligned}
$$

The total resistance is 120 Ω.

1.3.2 Roots

A *radical* , $\sqrt{\ }$, is a symbol we use to define the *root* of a number or an expression such as $\sqrt[3]{8}$, or say $\sqrt{a+b}$. A common misconception with the radical symbol is that some people refer to it as the *square root* symbol and automatically assume we want the square root which is not the case. We may want to find the 3^{rd} root as in $\sqrt[3]{8}$, possibly the $\frac{1}{2}$ root, or any n^{th} root.

There are three components to roots:

$$\sqrt[n]{a}$$

1. n is the degree of the root
2. a is the *radicand*.
3. the symbol, $\sqrt{\ }$, is called the *radical*

Most calculators have two buttons for defining roots of a number. One is the square root and is typically identified with the radical symbol while the other is for defining the n^{th} root, and is typically identified with a symbol that looks like $\sqrt[\square]{\ }$. There is one advanced calculator,TI-89, that doesn't have an n^{th} root button, and in those cases the user would have to convert the root to exponential form which is property 8 in the table of *Properties of Exponents*.

The n^{th} root of a number x, say the square root of x for this instance, is a number such that when squared will equal x. For example, the square root of 25, or $\sqrt{25}$, is 5 because $5^2 = 25$. Below are a few more examples.

Example 1.3.12:

$\sqrt{16} = 4$ (Square root of 16 equals 4) $\sqrt[3]{8} = 2$ (Cube root of 8 equals 2)

$\sqrt[4]{81} = 3$ (4^{th} root of 81 equals 3) $\sqrt[5]{32} = 2$ (5^{th} root of 32 equals 2)

Since there are no two negative numbers that when multiplied together give a negative, such as $(-a)(-a)$, then it is not possible to take the square root, or any even root, of a negative number with a result of real number. However, since $(-a)(-a)(-a) = -(a)^3$ such as $(-2)(-2)(-2) = -8$, then we can take the cube root, or any odd root, of an odd number.

Example 1.3.13:

$$\sqrt[3]{-8} = -2 \quad \text{since } (-2)^3 = -8 \qquad \sqrt[5]{-243} = -3 \quad \text{since } (-3)^5 = -243$$

Caution:
Note that $-a^2 \neq (-a)^2$. Recall the third exponential property $(ab)^n = a^n \cdot b^n$. Since $-a$ can be rewritten as $(-1)(a)$, then $-a^2 = (-1)(a)^2 = -a^2$; however, $(-a)^2 = (-a)(-a) = a^2$. This applies to any even exponent.

Properties of Roots

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $\left(\sqrt[n]{a}\right)^n = b$
4. $\sqrt[n]{a} = a^{\frac{1}{n}}$
5. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Example 1.3.14: – Apply property 1. to simplify the radical expression.

$$\begin{aligned} \sqrt{50} &= \sqrt{25 \cdot 2} \\ &= \sqrt{25} \cdot \sqrt{2} \\ &= 5\sqrt{2} \quad \sqrt{2} \text{ is an irrational number, so we leave it expressed this way for now.} \end{aligned}$$

Example 1.3.15: – Apply property 2. to simplify the radical expression.

$$\begin{aligned} \sqrt[3]{\frac{27}{8}} &= \frac{\sqrt[3]{27}}{\sqrt[3]{8}} \\ &= \frac{3}{2} \end{aligned}$$

Example 1.3.16: – Apply property 3. to simplify the radical expression.

$$\left(\sqrt[4]{7}\right)^4 = 7$$

Example 1.3.17: – Apply property 4. to simplify the radical expression.

$$\begin{aligned} \sqrt[3]{8} &= 8^{\frac{1}{3}} \\ &= 2 \end{aligned}$$

Example 1.3.18: –Apply property 5. to simplify the radical expression.

$$\begin{aligned}\sqrt[3]{8^2} &= 8^{\frac{2}{3}} \\ &= \left(8^{\frac{1}{3}}\right)^2 && \text{use exponential property } a^{mn} = (a^n)^m \\ &= 2^2 && \text{since } 8^{\frac{1}{3}} = 2 \\ &= 4\end{aligned}$$

More often than not, you will use a calculator to perform most calculations. However, for the next example we'll simplify the expression without the use of a calculator so that we can exercise different properties of exponents. In this section, most homework problems will specify whether or not to use a calculator. It's highly recommended that you follow directions as the exercises are intended to give you practice on manipulating expressions through the use of algebraic properties and exponential properties. After all, a calculator can only calculate numeric expressions, and you will encounter many expressions that are *alphanumeric*. **Alphanumeric** consists of letters (variables) and numbers.

Example 1.3.19:

Use the properties for both exponents and roots to simplify the following:

$$\sqrt[3]{\frac{27(.008)^2}{.027}}$$

Solution:

$$\begin{aligned}\sqrt[3]{\frac{27(.008)^2}{.027}} &= \sqrt[3]{\frac{3^3\left(\frac{8}{1000}\right)^2}{\frac{27}{1000}}} = \sqrt[3]{\frac{3^3\left(\frac{2^3}{10^3}\right)^2}{\frac{3^3}{10^3}}} \\ &= \sqrt[3]{\frac{3^3\left(\left(\frac{2}{10}\right)^3\right)^2}{\left(\frac{3}{10}\right)^3}} = \sqrt[3]{\frac{3^3\left(\left(\frac{1}{5}\right)^2\right)^3}{\left(\frac{3}{10}\right)^3}} \\ &= \sqrt[3]{\left(\frac{3\left(\frac{1}{5}\right)^2}{\frac{3}{10}}\right)^3} = \frac{3\left(\frac{1}{5}\right)^2}{\frac{3}{10}} \\ &= \frac{\frac{3}{25}}{\frac{3}{10}} = \frac{10(3)}{25(3)} \\ &= \frac{2}{5}\end{aligned}$$

1.4 Significant Digits and Rounding

1.4.1 Significant Digits

This section is primarily focused on **approximate numbers**; however, this implies there are **exact numbers** . An exact number results from either a definition, or from counting. For example, if we count the number of desks in a classroom, then there are an exact number of desks in the room. An approximate number is the result of a measurement. For example, if we measure the height of a tree, then there is no way to measure the exact height rather we have to settle for an approximation that is based upon how precise we want the measurement. The following are a few more examples of approximate versus exact numbers.

Example 1.4.1:

Exact:

1. The number of Skittles in a bag.
2. The number of eggs in a dozen.
3. The number of people at a football game
4. The number of grains of sand on the planet.

Approximate:

a. The distance across the United States.
b. The number of gallons of water in a specific swimming pool.
c. The weight of a specific individual person.
d. The height of the ceiling in the room you're standing in.

You will never find a measurement that will give you an exact number. All measurements are appoximate regardless of how sophisticated the measuring tool is. For example, If we measure the diameter of a golf ball with an analog set of dial calipers we would fine one answer. On the other hand, we would expect to find the diameter of the ball more precisely with a digital set. Regardless of the limitiation of the tools we use, there always exists a higher level of precision, and for this reason we can never know the exact diameter of the golf ball, or anything else we choose to measure.

Precision and Accuracy

- The **precision** of a number is indicated by the position of the last significant digit with respect to the decimal point.
- **Accuracy** refers to the number of siginificant digits.

Caution:
Not all decimal values indicate a measurement. For instance, 0.5 does not have to indicate a measurment, thus thought of as an approximate number. For example, 0.5 times the number of students in a class would indicate exactly half of the students in a class.

Both precision and accuracy rely on the concept of *significant digits* , so the following table outlines a few rules for determining if a digit is considered significant.

Rules for Determining Significant Digits

1. Any non-zero digit is significant.
2. All zeros between two nonzero digits are significant.
3. All zeros to the right of both the decimal point and the last nonzero digit are significant.
4. All leading zeros are not significant.
5. If a decimal is not present, then the following zeros after the last non-zero digit are not significant.

The following are examples of each of the rules just listed:

Example 1.4.2: – Examples for Rule 1.

All numbers are non-zero, thus are significant.

1.618	4 significant digits
9	1 significant digit
2.718281828	10 significant digits

Example 1.4.3: – Examples for Rule 2.

All zeros between two non-zero digits are significant

101	3 significant digits
0.120034	6 significant digits. The leading zero in front of the decimal is not significant.
3001.	4 significant digits

Example 1.4.4: – Examples for Rule 3.

All zeros to the right of both the decimal point and the last non-zero digit are significant.

3.12000	6 significant digits
5.00	3 significant digits. The two zeros at the end are defining the precision. If 5 was exact, then the two zeros are not significant, but that would have to be stated otherwise we assume not exact.

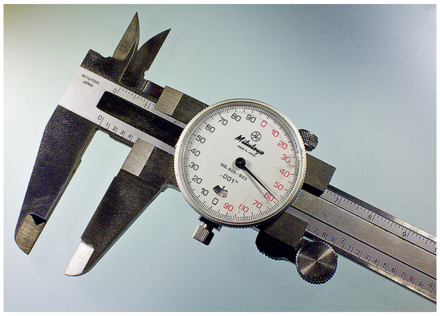


Figure 1.9: Analog Dial Caliper

Example 1.4.5: – Examples for Rule 4.

All leading zeros are not significant.

0.9	One significant digit
00.45	two significant digits
0.00078	two significant digits. The zeros are only used as a place holder for the decimal

Example 1.4.6: – Examples for Rule 5.

If a decimal is not present, then the following zeros after the last non-zero digit are not significant.

2000.	4 significant digits
2000	1 significant digit. Below is an explanation of the differences in greater detail

Discussion:
Lets take a closer look at the last example. The last example states that 2000 only has one significant digit while 2000. has four significant digits, so what is the difference? 2000 has only one significant digit, the non-zero digit of 2, because the measurement is stated to be accurate to the nearest thousand because the 2 is in the thousands place. However, the number 2000. with the decimal present at the end is defining the measurement to be accurate to the nearest ones place, thus making all four digits significant. To avoid this confusion, some texts will define a zero to be significant after a whole number only if has either a *tilde* (∼), or a *bar* (−) over the last significant digit. Property 2. says that all zero's between two significant digits are also significant, thus if we define 2000. as $200\bar{0}$, then they would both be defing a number that's accurate to the ones place with four significant digits.

Note: Significant figures are only relevant to numbers that are the result of a measurement. For example, a non-terminating decimal thats the result of a fraction such as $\frac{41}{11} = 3.7272\overline{72}$ does not have infinitly many significant digits.

A fair assumption when dealing with significant figures is that they represent the precision of the tool used during measuring. Most dial calipers, as shown in figure 1.9 (digital or analog), can measure with a precision of up to a thousandth of an inch, or 25 hundreths (.025) of a millimeter, so if we come across a measurement of 0.120 inches then we'd know that the tool used had a precision of a thousandth of an inch. On the other hand, if the measured value was 0.12 then we would assume that the tool used had a precision of hundredths of an inch.

1.4.2 Rounding Numbers

When working with approximate numbers, we have to round results often. Sometimes we need to round to the appropriate number of significant digits, and other times we have to round to the appropriate precision. In either of these cases, once we've located the last digit of the number to be rounded, say *a*, we look at the next digit to the right and if it has a value of five or greater then we round our previous digit we labeled *a* up to the next value. If the next digit was less than five, then we do nothing to *a*.

Example 1.4.7:

Round 3.1415927... to the ten thousandths place.

Solution:
Since the 5 is at the ten thousandths place (3.1415927), then we look at the next digit to the right of 5 (3.1415927) which is 9. Since $9 \geq 5$, then we round our 5 up to the next value which is 6. Our answer is 3.1416

When rounding to a certain number of significant digits, say *n*. We begin from the left and count the significant digits until we reach the desired number *n*.

Example 1.4.8:

Round 37.874606, 0.0032764, 97,134.4, and 58.00999 to 4 significant digits (*s.d.*), 3 *s.d.*, 2 *s.d.*, and 1 *s.d.*

Solution:

Number	4 *s.d.*	3 *s.d.*	2 *s.d.*	1 *s.d.*
37.874606	37.87	37.9	38	40
0.0032764	0.003276	0.00328	0.0033	0.003
97,134.4	97,130	97,100	97,000	100,000
58.00999	58.01	58.0	58	60

When rounding to the appropriate precision it is very much the same as when asked to round to a certain decimal place. Instead of being asked to round to say the third, or second decimal place, we would refer to those places by their respective place value names such as the thousandths place, or the hundredths place.

Example 1.4.9:

Round 37.874606, 0.0032764, 97,134.4, and 58.00999 to the tens, ones, hundredths, thousandths, and tenthousandths places.

Number	tens	ones	hudredths	thousandths	tenthousandths
37.874606	40	38	37.87	37.875	37.8746
0.0032764	0	0	0.0	0.003	0.0033
97,134.4	97,130	97,134	97,134.40	97,134.400	97,134.4000
58.00999	60	58	58.01	58.010	58.0100

1.4.3 Combining Approximate Numbers

Adding and Subtracting Approximate Numbers

Calculators have no way of determining whether a non-zero digit is significant, so a calculator will not be able to disinquish between 0.12 and 0.12000. It will be up to you to determine the correct place to round your result unless otherwise specified.

When adding or subtracting approximate numbers, the result can only be as precise as the least precise number. We do *not* round all our approximate numbers before combining them by any operation $(+,-,\times,\div)$, rather we round the final result to the least precise number when adding or subtracting. The process is a little different when multiplying and dividing.

Example 1.4.10:

Find the sum of 12.3 and 3.056 where both are approximate numbers and round your result to the appropriate precision.

Solution:

$$\begin{array}{r} 12.3 \\ +\ \ 3.056 \\ \hline 15.356 \end{array}$$

Since our least precise number is at the tenths place, our result must be rounded to the tenths place as well, thus our result is 15.4

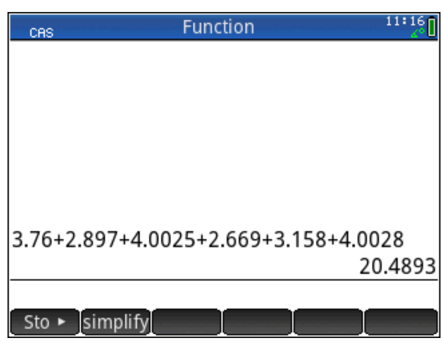
Adding too many numbers by hand becomes tedious, and relying on a calculator to perform those calculations is often preferable. For the purpose of this book I will illustrate all calculations using the HP Prime calculator; however, it is not necessary for anyone to go out and purchase one of these calculators. Though the HP Prime is loaded with features, I will only make use of features that are standard on most scientific calculators. The only reason I'm using the HP Prime for this text is because of the availablity of the emulator and its cost (free). At the time of this writing, the emulator for the HP Prime can be found at:
http://www.hp-prime.de/en/category/13-emulator.

Note: The specific buttons and their sequences used on the calculator will not be shown since both may differ depending on the calculator being used.

Example 1.4.11:

Find the sum of the following approximate numbers and round the result to the appropriate precision: 3.76, 2.897, 4.0025, 2.669, 3.158, 4.0028

Solution:
While we could certainly add these few numbers by hand; however, the use of a calculator will save us the tedious task.



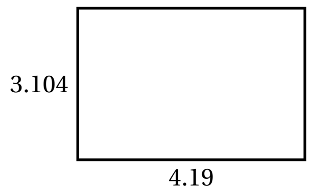
Since the least precise number (3.76) is approximated to the hundredths place, we must round our result to the hundredths place as well. Our final result is 20.49.

Multiplying and Dividing Approximate Numbers

When multiplying or dividing approximate numbers we must round our result to the least accurate number (least number of of significant digits). The reason for this is that the error becomes greater with respect to the exact value. For example, if a manufacturer produces a bolt or screw with a length that is designed to be 3/4 inches, the reality is that the bolt will never be exactly 3/4 inches. If the bolts produced are too long or too short with respect to the exact length of 3/4 inches then those bolts would be rejected. So, error is important and is discussed later in this section.

Example 1.4.12:

Calculate the area of the rectangle, and round the result to the appropriate number of significant digits.



Solution:
The area of a rectangle is calculated by multiplying the length, l, by the width, w. It does not matter which you choose for the length or width, so

$$
\begin{array}{r}
3.1\,0\,4 \\
\times \quad 4.1\,9 \\
\hline
2\,7\,9\,3\,6 \\
3\,1\,0\,4 \\
1\,2\,4\,1\,6 \\
\hline
1\,3.0\,0\,5\,7\,6
\end{array}
$$

Since the least *accurate* number, 4.19, has three significant digits, we must round our result to 3 significant digits as well. To do this we start from the left and work our way to the right until we have 3 significant digits. Our final answer is 13.0.

Caution:
Remember to never round intermediate steps to the least precise number or least accurate number until all arithmetic has been performed.

1.4.4 Understanding Error

The concept of error is important. In many cases, if the error in manufacturing an item is too large, then the item will be deemed defective and must be discarded, or recycled. Depending on the item being manufactured, the error may or may not be too strict; however, regardless of the strictness of the error the topic is important enough to know. Many students have already been exposed to the concept of error already, but are used to referring to it as a tolerance.

It is not possible to measure an item and determine the **exact value**, rather any measurement will only be an approximation. In order to determine the error of a measurement, we have to know the exact measurement. This exact number is only theoretical, and we use this to determine how far off the measurement is from the theoretical exact value, or just exact value.

There are three types of error. The first is the **absolute error** , and is defined as the approximated value minus the exact value. The absolute error can be either positive or negative depending whether the true value is greater or less than the approximated value.

The second type of error is called the **relative error** , where it is defined to be the ratio of the absolute error to the exact value. The third type of error is called **percent error** , and is the result of multiplying the relative error by 100 to give the percentage of the difference of the approximated value is to the exact value.

Types of Error

1. Absolute Error = Measured Value - Exact Value
2. Relative Error = $\dfrac{\text{Measured Value - Exact Value}}{\text{Exact Value}}$
3. Percent Error = Relative Error · 100%

Example 1.4.13:

A manufacturer has chosen a random beverage can from a lot of 1,000. The thickness of the can has been measured to be $39\mu m$, where the exact value is supposed to be $50\mu m$. If the percent error is greater than $\pm 17\%$, then the entire lot of 1,000 cans must be recycled. Determine the correct course of action for this lot of 1,000 cans.

Units measured in micrometers are identified by μm. One micrometer, or $1\mu m$, is 1 millionth of a meter, or 1×10^{-6} meters.

Solution:

$$\begin{aligned}
\text{Absolute Error} &= 39 - 50 \\
&= -11 \\
\text{Relative Error} &= \frac{-11}{50} \\
&= -0.22 \\
\text{Percent Error} &= -0.22 \cdot 100\% \\
&= -22\%
\end{aligned}$$

Since -22% is out of the error range, then the entire lot of 1,000 cans must be recycled.

Example 1.4.14:

In an attempt to expedite the process of determining which cans need to be recycled and those that don't in reference to example 1.4.13, what is the minimum and maximum thickness that a can must be to ensure the error is not smaller or larger than the required 17% ?

Solution:

To begin, we setup an expression using the appropriate formula. The formula that we need is either percent error, or relative error. Once set up, we should have only one variable that we must solve for which we'll denote as x.

$$\text{Relative Error} = \frac{\text{Measured Value - Exact Value}}{\text{Exact Value}}$$

Write down the correct formula to use

$$0.17 = \frac{x-50}{50}$$

Substitute known values. Since all units are the same, it is not necessary to write them down while solving.

$$50(0.17) = x - 50$$

Multiply both sides of the equation by 50.

$$8.5 + 50 = x$$

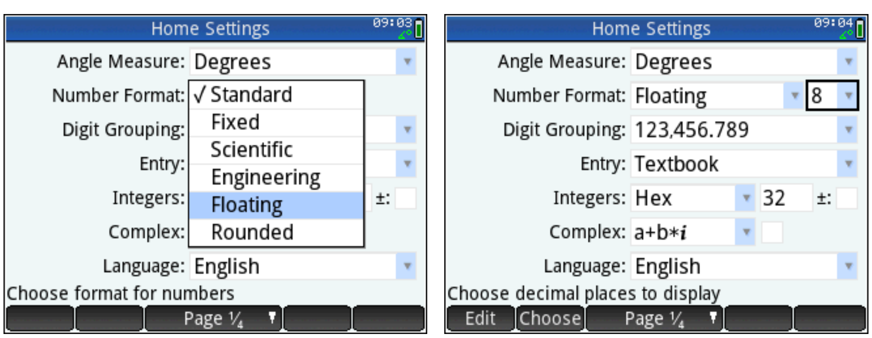
Add 50 to both sides of the equation.

$$x = 58.5\mu m$$

If the thickness of the can is greater than $58.5\mu m$ then the lot must be recycled. Similarly, we can find the lower limit by setting the relative error equal to -0.17 and solving again. If done following the same procedure, you will find that the lower bound is $41.5\mu m$.

1.4.5 Setting Precision on the Calculator

Though the examples in this book are not too tedious, certain calculations where there are a lot of values that must by added can be. For this reason it's helpful to set the precision in the calculator. By default, most calculators will display around ten digits; and, more ore often than not, this is more than what's needed. You can change the number of decimal places by use of the *mode* button on most calculators. Other calculators may be different, and in those instances you should refer to the operators manual. For instance, to set the number of decimals, also known as the floating point, on the HP Prime calculator used in this text, you must press shift, and then the home button to access the settings. On the first page in the settings menu under *Number Format*, there is a list of available options for displaying numbers as shown in figure 1.10. To set the number of visible decimal values choose the option labeled *Floating*. Once selected, a new dropdown menu appears beside the number format setting allowing you to choose which value you want.



(a) First (b) Second

Figure 1.10: HP Prime Settings

1.5 Scientific and Engineering Notation

Both scientific notation, and engineering notation allow us to deal with very large and very small numbers in a less cumbersome manner. For example, astronomers have to deal with numbers that get incredibly large such as the distance to the sun which is about 92,960,000 miles, or 149,600,000 kilometers. The mass of the sun is 1,988,435,000,000,000,000,000,000,000,000 kg. Other than the fact that dealing with numbers such as these are very cumbersome, attempting to use such large numbers in decimal form invites plenty of room for mistakes to occur. Another advantage to using either scientific or engineering notation is that they both allow calculators to display large and small numbers on a screen with limited space.

In astronomy, it is commton to see distances measured in *astronomical units*, *au*, where *1au* is the distance from the earth to the sun.

1.5.1 Scientific Notation

Scientific notation is of the form $a \times 10^n$, where a is the first non-zero digit in the number and n is an integer. If the number is greater than 1, or less than -1, then n will represent the number of places that the decimal had to move to be placed behind the first non-zero digit. For example, the first non-zero digit in the value for the mass of the sun is 1, and the decimal is behind the last digit of zero, thus the decimal would have to be moved 30 places to the left. So, in scientific notation the mass of the sun is 1.988435×10^{30}. If the decimal has to be moved to the right then n will be negative. This occurs anytime the value of the number is a fraction, or greater than -1, but less than 1. For example, the lightest element in the periodic table is hydrogen which weighs .0000899 grams per cubic centimeter. In scientific notation hydrogen weighs $8.99 \times 10^{-5} g/cm^3$ The following are a couple more example numbers rewritten in scientific notation.

If a number is less than 10, such as numbers 1 - 9, it isn't necessary to rewrite it in scientific notation. For instance, 9 in scientific notation would be 9×10^0, and since $10^0 = 1$, this is just another way of stating 9×1. However, when combining numbers in scientific notation, you may find it helpful to write numbers from 1 - 9 in scientific notation anyway to avoid making mistakes when applying the properties of exponents.

Example 1.5.1:

$$\underbrace{149,600,000}_{\text{8 decimal places}}\ km = 1.496 \times 10^{8}\ km \quad , \quad \underbrace{.000000267}_{\text{7 decimal places}} = 2.67 \times 10^{-7}$$

All significant digits should be included when rewriting a number in scientific notation as shown in example 1.5.2, and any zeros present in scientific notation should be considered significant.

Example 1.5.2:

$$0.000350 = 3.50 \times 10^{-4} \tag{1.1}$$

$$3420\bar{0}000 = 3.4200 \times 10^{7} \tag{1.2}$$

Recall that zeros with a line over them, or a tilde are labelled significant. These rules can be reviewed in section 1.1.

We refer to a number that has not been compressed, as in scientific notation, as the **ordinary** notation of a number. Reversing the process is all that needs to be done to put a number in ordinary form.

1.5.2 Engineering Notation

Engineering notation is similar to scientific notation except that the exponent, n, in the expression $a \times 10^n$ must be divisible by 3, and a is a number that is greater than or equal 1 but less than 1000, or $1 \leq a < 1000$. The main motivation for using engineering notation is because it is used to express quantities in terms of the *International System of Units*, or *SI*. For example, we've just seen the micro meter used in example 1.4.13, where *micro* references millionths; thus $3\mu m = 0.000003m$, or in engineering notation we have $3 \times 10^{-6}m$. Notice that exponent, -6, is divisible by 3. Engineering notation is commtonly used to represent 16 of the 20 *SI* prefixes such as *kilo*, *mega*, and *nano* to name just a few. Those 16 out of 20 prefixes are all multiples of 1000=1×10^3. This is the reason for having the exponent of 10 divisible by 3. Since *SI* units are not the focus of this section, we will come back to them in a later chapter.

Example 1.5.3:

Rewrite $5\mu m$ in engineering notation.

Solution:
Since $5\mu m$ is equal to $0.000005m$ then in engineering notation this would be $5 \times 10^{-6}m$.

Notice that 5 is greater than 1 and less than 999; and the exponent is a multiple of 3.

Example 1.5.4:

Express the following in engineering notation:

1. 3,141,500
2. 1.618
3. 270,000,000,000
4. 0.000 000 09

Solution:

1. $3,141,500 = 3.1415 \times 10^6$
2. $1.618 = 1.618 \times 10^0$
3. $270{,}000{,}000{,}000 = 270 \times 10^9$
4. $0.000\,000\,09 = 90 \times 10^{-9}$

1.5.3 Scientific and Engineering Notation on Calculators

Most scientific calculators today have the ability to display results in either scientific notation, or engineering notation; however, some calculators may differ on how the answer is displayed. There are basically two methods that calculators use to identify the $\times 10^n$ part. One method is to display the number exactly the same way we've shown here except the $\times 10^n$ part is written smaller. For example, 3.1415×10^6 might be displayed on the calculator as $3.1415_{\times 10}{}^{06}$. The other method, much more commton, lets the letter **E** represent $\times 10^n$. For example, the output on the screen would look similar to 3.1415 E 6. Each calulator will differ on how to set the mode, so you will likely have to read the documentation that came with your calculator to see how this is done. Usually scientific notation is abbreviated as SCI and engineering notation as ENG.

Once you've set the calculator to display in either mode, you can use the calculator to convert numbers to the mode you're in. This may prove helpful until you're proficient enough to make the conversions on your own. Eventually, you will recognize 5 E 6 as synonymous as having it spelled out as 5 million.

To change the display modes on the HP Prime we press the *shift* key then the home button to access the home settings. On the first page under *Number Format* we can change the mode to engineering as shown below.

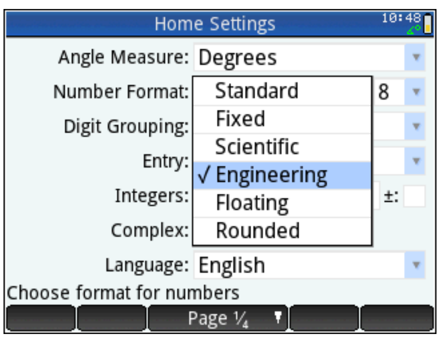


Figure 1.11: Home Settings

Example 1.5.5:

Use your calculator to rewrite 27,000,000,000 in both engineering, and scientific notation.

Solution: We will begin by setting the calculator mode to engineering notation. Once this is done we only need to input 27,000,000,000 in the calculator and hit Enter or the equal button. The output for engineering and scientific are:

Engineering : 27 E 9

Scientific : 2.7 E 10

Depending on how many many fixed digits are set in the calculator, your answers could look similar to 27.0000000 E 9.

To enter numbers in either mode, we use the button that either looks like **EE**, or **EEX** to denote $\times 10$. For example, to enter 0.0000000735 in engineering notation we would enter the following sequence of buttons on the calculator:

73.5 **EEX** +/- 9

To assign a number to be negative on the calculator, we use the buttons that either look like +/-, or more commonly the button that appears like (-).

Try this on your own to make sure that the output on your calculator is what's excpected.

1.5.4 Products and Quotients of Scientific or Engineering Notation

When working with very large, or very small numbers, the calculations can be made a little easier in scientific notation.

Example 1.5.6:

Perform the following operation in scientific notation.

$$64,000,000,000,000 \times 23,000,000$$

Solution:
We begin by rewriting the expression in scientific notation. Once that's complete, we take advantage of the rules for exponents to simplify the expression.

$$\begin{aligned}
64,000,000,000,000 \times 23,000,000 &= (6.4 \times 10^{13})(2.3 \times 10^{7}) \\
&= 6.4 \times 2.3 \times 10^{13} \times 10^{7} && \text{reorder since everything is multiplied} \\
&= 14.72 \times (10^{13} \times 10^{7}) && \text{multiply 6.4(2.3) to get 14.72} \\
&= 14.72 \times 10^{13+7} && 10^{13} \times 10^{7} = 10^{13+7} \\
&= 14.72 \times 10^{20} && \text{14.72 is not in engineering notation} \\
&= (1.472 \times 10^{1}) \times 10^{20} && \text{convert 14.72 into Scientific notation} \\
&= 1.472 \times 10^{21} && 10^{1} \times 10^{20} = 10^{1+20}
\end{aligned}$$

Recall from section 1.3 that $\dfrac{a^n}{a^m} = a^{n-m}$. Use this property in conjuntion with others to perform the operations in the following example.

Example 1.5.7:

Perform the following operations in engineering notation. Round result to three decimal places.

$$\frac{0.000045 \cdot 21,000,000}{2.718000}$$

Solution: To begin, rewrite all numbers in engineering notation, then use the properties of exponents to combine like bases. In the event that we have to perform any intermediate rounding, which is rounding that occurs before we get to the final answer, we will round to two more decimal places than what's required. This is good practice to help avoid compounding the error too much before reaching the final result.

$$\begin{aligned}
\frac{0.000045 \cdot 21{,}000{,}000}{2.718000} &= \frac{(45 \times 10^{-6})(21 \times 10^{6})}{(2.718 \times 10^{0})} && \text{note: the parenthesis here are not necessary, but help us identify each number} \\
&= \frac{45 \times 21}{2.718} \times \frac{10^{-6} \cdot 10^{6}}{10^{0}} && \text{reorder in terms of like numbers} \\
&= 347.68212 \times 10^{-6+6-0} && \text{perform the arithmetic and apply properties of exponents.} \\
&= 347.682 && \text{not necessary to put } 347.682 \times 10^{0} \text{ since } 10^{0} = 1, \text{ though not incorrect if you do.}
\end{aligned}$$

It becomes tedious to perform larger calcualtions by hand. If your calculator is in the desired mode, such as engineering, then it's not necessary to enter numbers in that manner; rather the main benefit to doing so cuts down on the number of key strokes required to enter the numbers.

Note: Arrow keys such as [⇒], [⇐], [⇑], and [⇓] reference the direction keys on your calculator. Depending on the calculator you're using, it is sometimes necessary to use these arrow keys to move the cursor in or out of the correct position.

Example 1.5.8:

Perform all arithmetic of the follwing using a scientific calculator. The result should be in engineering notation.

$$\frac{0.000\,001\,7}{0.000\,097\,3} \cdot 12{,}000$$

Solution:
As mentioned above, it is not necessary to enter the values in the calculator in engineering notation, but it will cut down on the keystrokes required. For this reason, and to avoid entering all those zeros without a mistake, we'll convert each number to engineering notation first before entering them in the calculator.

$$\frac{1.7\,\mathrm{E}{-6}}{97.3\,\mathrm{E}{-6}} \cdot (12\,\mathrm{E}\,3)$$

Now that we have our values converted to the appropriate notation, the following are the keystrokes required to enter in the calculator. Keep in mind that since everything is being multiplied or divided (nothing added or subtracted, or any other operations), it is not necessary to define the numerator and the denominator in parenthesis.

1.7 [**EEX**] [+/-] 6 [÷] 97.3 [**EEX**] [+/-] 6 [⇒] [×] 12 [**EEX**] 3

If entered correctly, the result should appear like the window in figure 1.12:

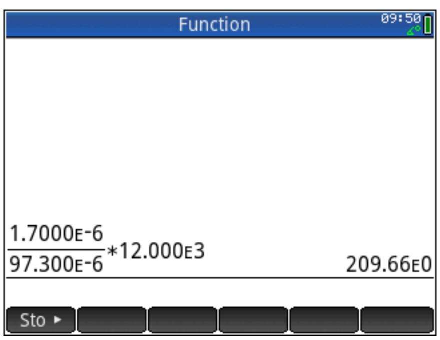


Figure 1.12: Home Settings

Example 1.5.9:

According to npr.org, there are multiple stars in the universe for every grain of sand on the entire earth. The number of grains of sand on earth has been roughly approximatted to be 7.5×10^{18} grains. Using this approximation of grains of sand, how many stars are there in the universe if we assume that there are at least 2 stars for every grain of sand? Leave the result in engineering notation.

Solution: We need to multiply 7.5×10^{18} times 2 since there are 2 stars per grain of sand. Without using a calculator we would convert the number 2 to engineering notation first before multiplying.

$$(7.5 \times 10^{18})(2 \times 10^{0}) = 7.5(2) \times 10^{18+0}$$ rewrite the number 2 in engineering notation, and reorder

$$= 15 \times 10^{18}$$

Based upon this huge assumption, there are at least 15×10^{18} stars in the universe.

Figure 1.13: Stars

Example 1.5.10:

Light travels at approximately 300 000 *km/s*. How long in minutes does the suns light take to travel to earth if the earth is 150 000 000 *km* away?

Solution:

First, we convert each value to engineering notation:

$$300,000 km/s = 300 \times 10^{3} km/s$$
$$150,000,000 km = 150 \times 10^{6} km$$

To find either speed (s), time (t), or distance (d) we generally use the formula $d = s \cdot t$, or distance is equal to speed times time and solve for the variable we need. Since we want to find the time (how long it takes) we rewrite the equation as $t = \frac{d}{s}$.

$$\frac{d}{s} = \frac{150 \times 10^{6}\ km}{300 \times 10^{3}\ km/s} = \frac{150}{300} \times \frac{10^{6}}{10^{3}} \times \frac{km \cdot s}{km} \qquad \frac{km}{\frac{km}{s}} = \frac{km}{1}\frac{s}{km} = \frac{km \cdot s}{km}$$

$$= \frac{1}{2} \times 10^{6-3}\ s$$

$$= 500\ s$$

Now that we know how many seconds it takes, we just need to convert 500 s into minutes by dividing by 60 seconds.

$$\frac{500\ s}{60\ s} \approx 8.33m$$

1.6 Addition and Subtraction of Expressions

1.6.1 Variables

A **variable** is any symbol used to represent an unknown quantity. Usually we use letters such as *n, x,* or *b*. Often we find ourselves solving for one, or more, of these variables, but that is not the focus of this section.

There are certain letters that have been designated for a specific use that we don't consider to be a variable, rather we call them **constants** . For example, the greek letter π is assigned to represent the ratio of the the circumference of a circle to the diamter which is approximately 3.1415. Another example would be the letter $e = 2.71828$. There are several letters assigned specific values that we avoid reassigning values to. On occasion, some algebraic expressions require so many variables that it makes it tedious to keep up with, so we introduce the concept of **subscripts** . A subscript is a small number, or letter, written to the bottom right of a variable such as a_1, a_2, and a_3. By introdcuing the subscript, we have effectively increased, dramatically, the number of variables that are available to us to use.

Note: The subscript is a counter

$$a_1 = \text{the first } a$$
$$a_2 = \text{the second } a$$
$$\vdots$$

it is not a math operation such a squaring or cubing.

More often than not, when learning a new algorithm, or equation, it's common to make use of variables to introduce these new algorithms because it is the most general expression we can get that applies to all values unless otherwise stated. For example, $\sqrt{n}\,, n \geq 0$ means that n can be any real number as long as n is greater than or equal to zero. In example 1.6.1 in the margin, the distributive law is illustrated where *a, b,* and *c* can be any real number.

Example 1.6.1:

$$a\,(b+c) = a \times b + a \times c$$

1.6.2 Algebraic Expressions

The term **algebraic expression** is used to describe any combination of variables and/or constants just as we saw in example 1.6.1. Below are some other examples of algebraic examples.

Example 1.6.2:

$$a^2 + b^2 - c^2 \tag{1.3}$$

$$ax^2 + bx + c \tag{1.4}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.5}$$

1.6.3 Factors of Algebraic Expressions

Factors of an algebraic expression are any terms or numbers that can be multiplied together to get the original expression. For example, $5x^2$ is a **term** and $5 \cdot x \cdot x = 5x^2$ so x is a factor of $5x^2$ and 5 is a factor of $5x^2$. Another way to think of a factor is anything that divides the expression evenly. For instance, 7 is a factor of 21 since $\dfrac{21}{7} = 3$.

Factors of an algebraic expression should not be confused with terms of the expression. Every algrabric expression has at least one factor namely itself. For instance, the expression a^2 has one term, but has two factors: a, and a^2.

$3a(x^2-4)+5b(x^2+4$ is an expression with terms $3a(x^2-4)$ and $5b(x^2+4)$ and factors of 3, a, (x^2-4), 5, b, and (x^2+4) as well as the products of these factors. Since (x^2-4) can be factored into $(x-2)(x+2)$, then $(x-2)$ and $(x+2)$ are also factors. (x^2+4) can not be factored with any real results.

1.6.4 Monomials and Multinomials

An algebraic section with only one term is called **monomial** . An algebraic expression with two terms is called **binomial** , and an expression with three terms is called **trinomial**. Any algebraic expression that exceeds one term is described as **multinomial** . Though binomials and trinomials are also multinomials, we typically refer to multinomials as algebraic expressions that exceed three terms, but stating a binomial or trinomial as multinomial is not wrong.

Example 1.6.3:

1. $3x^2$ is called monomial since it only has one term
2. b^2-4ac is called binomial since it has two terms: b^2, and $-4ac$
3. ax^2+bx+c is trinomial.
4. $x^3+3x^2y+3xy^2+y^3$ is multinomial since it has more than three terms.

A **polynomial** is an algebraic expression with only nonnegative integer exponents. The **degree** of a polynomial is defined by the largest exponent of all the terms of the expression. For example, $2x+4x^2+7$ is a polynomial of degree 7 since the largest exponent is 2 and all the exponents are nonnegative integers. Below are some more examples of exponents and their degree:

Example 1.6.4:

1. $5x^2$ is a polynomial of degree 2
2. $7x^6-1$ is a polynomial of degree 6
3. $3x^2-2x^7+12x^4$ is a polynomial of degree 7

A **numerical coefficient** , or just **coefficient** , is the number that the term is multiplied by. Notice in part 1. of example 1.6.4 above that the term is being multiplied by 5, thus the coefficient is 5. The coefficient in part 2. is 2, and in part 3 we have several terms with coefficients where the coefficient for $3x^2$ is 3, the coefficient of the leading term $-2x^7$ is -2, and finally the coefficient of $12x^4$ is 12.

1.6.5 Like Terms

Some terms in an algebraic expression have the same variables and powers that possibly differ only by the coefficient which we call **like terms** . For instance, $2a+3a+4b$ has two like terms: $2a$ and $3a$. Since these are like terms we can combine them by adding their coefficients $2a+3a=(2+3)a=5a$. It's important to note that only the variables and exponents matter when identifying like terms. For example, $9x^3yz$ and $-12x^3yz$ are like terms because the only thing that differs between them is the numerical coefficient. On the other hand, x^3y^2z and x^2y^3z are not like terms because the exponents on the variables are different.

You will encounter many algebraic expressions throughout this book that appear at first glance to be daunting; however, identifying like terms and combining them will help transform the expression into what is called a **simplified** expression. A simplified expression will only contain terms that have no longer have similar terms.

1.6.6 Combining Multinomials

In order to combine monomials, we only need to either add or subtract the coefficients of similar terms.

Example 1.6.5:

$$
\begin{aligned}
&(2x^2 - xy + 9y^2) + (5x^2 - 7xy - 7y^2) \\
&\qquad = (2x^2 + 5x^2) + (-xy - 7xy) + (9y^2 - 7y^2) && \text{Reorder similar, or like, terms} \\
&\qquad = 7x^2 - 8xy + 2y^2 && \text{Combine like terms until there are no similar terms left}
\end{aligned}
$$

Example 1.6.6:

$$
\begin{aligned}
&(3a + 2ab - 3c) - (4ab + 7c - a) \\
&\qquad = (3a - (-a)) + (2ab - 4ab) + (-3c - 7c) && \text{Reorder like terms, but don't forget to distribute the negative sign in front of the second grouping in parenthesis} \\
&\qquad = 4a - 2ab - 10c
\end{aligned}
$$

There are several ways of grouping terms together. The most common method is to make use of parenthesis, ( ), to group terms; however, some texts will use brackets, { }, braces, [ ], along with parenthesis. In this text, we will use nested parentheses as shown in the next example.

Example 1.6.7:

$5-(2+3a(c+b))-4c(a-b)$

$= 5-(2+3ac+3ab)-4ac+4bc$	Begin with the innermost parentheses first and work outward
$= 5-2-3ac-3ab-4ac+4bc$	Distribute the negative throughout the last set of parentheses only after all inner parentheses have been simplified
$= 3-3ab-7ac+4bc$	Combine like terms. The order in the final result does not matter.

1.7 Multiplication of Algebraic Expressions

1.7.1 Multiplying Monomials

When multiplying monomials we begin by multiplying the coefficients. After the coefficients have been multiplied we can multiply the remaining factors of the monomials following the properties of exponents from section **??**.

Example 1.7.1:

1) $(2xy)(4x^2y) = 8x^{1+2}y^{1+1} = 8x^3y^2$

2) $(-5a^3b^2)(-3a^{11}b^5) = (-5\cdot(-3))a^{3+11}b^{2+5} = 15a^{14}b^7$

3) $(4x^2yz^3)(-3xy^5z^6) = -12x^{2+1}y^{1+5}z^{3+6} = -12x^3y^6z^9$

Below is an example of the distributive property applied with multiple terms inside the parenthesis.

$$a\,(b+c+d) = ab + ac + ad$$

To multiply a monomial with a multinomial you need to make use of the distributive property. Recall from section 1.2 that the distributive property takes the value outside the parenthesis is multiplied throughout all terms inside the parenthesis.

Example 1.7.2:

$$\begin{aligned} -3x^2y^3(2x-5y^2+3xy) &= -6x^{2+1}y^{3+0} + 15x^{2+0}y^{3+2} - 9x^{2+1}y^{3+1} \\ &= -6x^3y^3 + 15x^2y^5 - 9x^3y^4 \end{aligned}$$

Notice that there are no like terms, so this is as far as we go

1.7.2 Multiplying Multinomials

When multiplying two multinomials together, we need to multiply each term in the first multinomial by each term in the second multinomial.

Example 1.7.3:

Expand the following by multiplying the two multinomials:

$$(a+b+c)(x+y+z)$$

To begin, we multiply the first term in the first multinomial, a, by every term in the second multinomial.

$$(a+b+c)(x+y+z) = ax + ay + az + (b+c)(x+y+z)$$

Next, take the second term in the first multinomial, b, and multiply it by every term in the second monomial.

$$ax + ay + az + (b+c)(x+y+z) = ax + ay + az + bx + by + bz + c(x+y+z)$$

Lastly, we distribute c throughout the second multinomial to get our result.

$$ax + ay + az + bx + by + bz + cx + cy + cz$$

Notice in example 1.7.3 that there were a total of 9 terms in the result. This will always happen; however, there are instances where some terms are similiar and can be combined.

Example 1.7.4:

Multiply the following two multinomials together:

$$(2x^3 + 5 - 6y^2)(-3x + 2y - 7z^4)$$

Solution:

We first begin by multiplying $2x^3$ by every term in the second multinomial:

$$(2x^3+5-6y^2)(-3x+2y-7z^4) = -6x^4+4x^3y-14x^3z^4+(5-6y^2)(-3x+2y-7z^4)$$

Next, we multiply 5 by every term in the second multinomial:

$$\begin{aligned} &-6x^4 + 4x^3y - 14x^3z^4 + (5 - 6y^2)(-3x + 2y - 7z^4) \\ &= -6x^4 + 4x^3y - 14x^3z^4 - 15x + 10y - 35z^4 - 6y^2(-3x + 2y - 7z^4) \end{aligned}$$

Lastly, we distribute $-6y^2$ throughout the multinomial. Notice that the minus sign goes with the term $-6y^2$.

$$\begin{aligned} &-6x^4 + 4x^3y - 14x^3z^4 - 15x + 10y - 35z^4 - 6y^2(-3x + 2y - 7z^4) \\ &= -6x^4 + 4x^3y - 14x^3z^4 - 15x + 10y - 35z^4 + 18xy^2 - 12y^3 + 42y^2z^4 \end{aligned}$$

In this example, there are no like terms, thus we can not simplify further.

1.7.3 Difference of Squares

When multiplying two binomials where one term in each is the same, and the other term in each differ only by a sign then the result will be a **difference of squares** as shown below.

$$(a+b)(a-b)=a^2-ab+ab-b^2=a^2-b^2$$

This special product appears often throughout algebra, and is beneficial to recognize the difference of squares when you encounter it.

Example 1.7.5:

1) $(x+3)(x-3)=x^2-\cancel{3x}+\cancel{3x}+9=x^2-9$ $\quad 9=3^2$

2) $(2x-5)(2x+5)=4x^2+\cancel{10x}-\cancel{10x}-25=4x^2-25$ $\quad 4x^2=(2x)^2$

Notice in the example above that the two middle terms always cancel out. This will always occur with the difference of squares.

1.7.4 Squaring Binomials

The square of a binomial is of the form $(a+b)^2$, and the purpose of the exponent is the same as any other time we square a number; we multiply the binomial by itself.

$$(a+b)^2=(a+b)(a+b)$$

When performing this operation, recall the procedure for multiplying multinomials earlier in this section. The procedure is the same, mulitplying every term in the first polynomial by every term in the second.

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2=a^2+2ab+b^2$$

$$(a-b)^2=(a-b)(a-b)=a^2-ab-ba+b^2=a^2-2ab+b^2$$

Example 1.7.6:

1) $(x+2)^2=(x+2)(x+2)=x^2+2x+2x+4=x^2+4x+4$

2) $(3L-4)^2=(3L-4)(3L-4)=9L^2-12L-12L+16=9L^2-24L+16$ $\quad (3L)^2=3^2L^2=9L^2$

3) $(x+h)^2=(x+h)(x+h)=x^2+xh+xh+h^2=x^2+2xh+h^2$

4) $\left(\frac{1}{3}-y\right)^2=\left(\frac{1}{3}-y\right)\left(\frac{1}{3}-y\right)=\left(\frac{1}{3}\right)^2-\frac{1}{3}y-\frac{1}{3}y+y^2=\frac{1}{9}-\frac{2}{3}y+y^2$

1.8 Division of Algebraic Expressions

We must be careful when writing or rewriting algebraic expressions. Recall that the division can be represented in several ways such as

$$\frac{1}{b}(a), \quad a \div b, \quad \frac{a}{b}, \text{ and } \quad a/b$$

all mean the same thing.

Caution: Remember that the numerator and the denominator of a fraction represent grouping symbols, so $a+b/c+d$ is not the same as $\frac{a+b}{c+d}$; rather $a+b/c+d = \frac{ac+b+dc}{c}$.

There are a couple ways of representing algebraic expressions where one is what is best described as "in-line" (see below). Care must be taken when writing algebraic expressions, especially those involving fractions, in-line because of grouping. Parentheses are used extensively to denote terms grouped together. The second is commonly known as "display mode" where certain aspects of the algebraic expression such as fractions are identified without parentheses. Below is an example of both ways.

$$\text{in-line}: \ (a+b)/(c+d), \quad \text{display mode}: \ \frac{a+b}{c+d}$$

Some may argue that there isn't really a difference; however, due to so many misinterpretations and mistakes it is important to mention.

1.8.1 Division of Two Monomials

To divide one monomial with another, or in other words to perform the quotient of two monomials, we will make use of the properties of exponents that was introduced in section 1.3.1. Recall that $\frac{a^m}{a^n} = a^{m-n}$. When dividing two monomials we will simplify each factor of each term with this property.

$$\frac{3x^4y^2}{6x^3y^5} = \frac{3}{6}x^{4-3}y^{2-5} = \frac{1}{2}xy^{-3}$$

Often results will be asked to be put shown with positive exponents only, thus we again use the properties of exponents to rewrite $\frac{1}{2}xy^{-3}$ as $\frac{x}{2y^3}$

Example 1.8.1:

Divide $9x^3y^2z^{-3}$ by $3xy^4z^{-6}$ with a result of only positive exponents.

Solution:
Be careful when using the rule $\frac{a^m}{a^n} = a^{m-n}$ when there are negative exponents.

$$\begin{aligned}
\left(9x^3y^2z^{-3}\right) \div \left(3xy^4z^{-6}\right) &= \frac{9x^3y^2z^{-3}}{3xy^4z^{-6}} \\
&= \frac{9}{3}x^{3-1}y^{2-4}z^{-3-(-6)} \\
&= 3x^2y^{-2}z^3 \\
&= \frac{3x^2z^3}{y^2}
\end{aligned}$$

Example 1.8.2:

Divide $6a^{-4}b^3c^{-5}$ by $42a^{-2}b^3c$ with a result of only positive exponents.

$$\begin{aligned}
\left(6a^{-4}b^3c^{-5}\right) \div \left(42a^{-2}b^3c\right) &= \frac{6a^{-4}b^3c^{-5}}{42a^{-2}b^3c} \\
&= \frac{6}{42}a^{-4-(-2)}b^{1-1}c^{-5-1} \\
&= \frac{a^{-2}b^0c^{-6}}{7} \\
&= \frac{1}{7a^2c^6} \qquad \text{recall } b^0 = 1
\end{aligned}$$

1.8.2 Dividing Multinomials by a Monomial

Recall that when adding fractions that the denominator must be the same. Once you've found a common denominator, then all there is left to do is add the numerators while the denominator remains the same.
For example, $\frac{2}{3}+\frac{5}{3} = \frac{2+5}{3} = \frac{7}{3}$. The same is true for all fractions, even those where the numerator and denominator are multinomials or monomials. In addition, you can also reverse the last step and break up a fraction into the sum of two or more fractions all with the same denomintor as shown in the margin.

Note: Fractions can be broken up into the sum of multiple fraction all with the same denominator:

$$\frac{a+b+c}{n} = \frac{a}{n}+\frac{b}{n}+\frac{c}{n}$$

To divide a multinomial by a monomial, the expression needs to be broken up into the sum of multiple fractions as shown in the margin.

Example 1.8.3:

Divide $(6x^2y^3+8x^3z)$ by $(2xy^3z^2)$

Solution:

$$(6x^2y^3+8x^3z) \div (2xy^3z^2) = \frac{6x^2y^3+8x^3z}{2xy^3z^2}$$

Notice that the parenthesis in the numerator and denominator are no longer present, but are implied because a fraction is a grouping symbol.

$$\begin{aligned}
&= \frac{6x^2y^3}{2xy^3z^2}+\frac{8x^3z}{2xy^3z^2} \\
&= 3x^{2-1}y^{3-3}z^{0-2}+4x^{3-1}y^{0-3}z^{1-2} \\
&= 3xz^{-2}+4x^2y^{-3}z^{-1}
\end{aligned}$$

This is technically the solution; however, if we want positive exponents then, we go one step further.

$$= \frac{3x}{z^2}+\frac{4x^2}{y^3z}$$

Example 1.8.4:

Divide $a^2 - b^2 + 4ac + c^2$ by $3ac$. Simplify the result using only postive exponents.

Solution:

$$\begin{aligned}
(a^2 - b^2 + 4ac + c^2) \div (3ac) &= \frac{a^2 - b^2 + 4ac + c^2}{3ac} \\
&= \frac{a^2}{3ac} - \frac{b^2}{3ac} + \frac{4ac}{3ac} + \frac{c^2}{3ac} \\
&= \frac{1}{3}a^{2-1}c^{-1} - \frac{b^2}{3ac} + \frac{4\cancel{ac}}{3\cancel{ac}} + \frac{1}{3}a^{-1}c^{2-1} \\
&= \frac{a}{3c} - \frac{b^2}{3ac} + \frac{4}{3} + \frac{c}{3a}
\end{aligned}$$

Notice in the last example that its easier, or quicker, to cancel the terms out that are exactly alike as opposed to rewriting the exponent as the difference of the numerator to the denominator.

1.8.3 Dividing a Multinomial by a Multinomial

When dividing a Multinomial by another multinomial the process is much different dividing by just a monomial. To begin, the instructions are for dividing a polynomial by another polynomial. Recall that a polynomial is of the form:

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots a_2x^2 + a_1x + a_0, \quad \{a_0, \cdots, a\} \in \mathbb{R}, \text{ and } n \in \mathbb{Z}$$

The subscripts and the superscripts in this definition of a polynomial are not intended to imply that if $a = 5$ then $a_{n-1} = 4$; rather they are only put there as a way of saying that the coeffecients are different, or possibly the same. The same goes for the exponents except the terms are written in decreasing order and are non-negative. Below is an example of a polynomial.

$$3x^7 - 6x^4 + 3$$

Review section 1.6.4 for examples of polynomials.

The process to dividing one polynomial by another is the same as when we performed numerical long division except now we have variables, but the process is the same. The process is probably best described with an example, so the following is an example broken down into steps.

Divide $-7x^2 + 6x^3 + 10x - 4$ by $3x^2 + 4 - 2x$

Step 1: Write the dividend and the divisor in decreasing order.

Since the largest exponent in the dividend 3, then that will be the first term then term with 2 and so on. We have to do the same to the divisor thus we have the following:

$$(6x^3 - 7x^2 + x - 4) \div (3x^2 - 2x + 4)$$

To perform long division, we will rewrite the problem with a division symbol just as we do with numerical long division.

$$3x^2 - 2x + 4 \,\big)\overline{\;6x^3 - 7x^2 \quad + x - 4\;}$$

If there are any missing terms, in this case there are none, then it helps to write them in any way with a coefficient of zero. The purpose of this is to keep things nicely ordered, but is not necessary.

Step 2: Divide the leading term in the dividend by the leading term in the divisor.

To begin the process of division, we divide the leading (first) term in the dividend , $6x^3 - 7x^2 + x - 4$, by the leading term in the divisor, $3x^2 - 2x + 4$.

$$\frac{6x^3}{3x^2} = 2x$$

Another way to think of it, which may prove to be faster, is to determine what needs to be multiplied to $3x^2$ to get $6x^3$? The result is the same of course , $2x$, but the process may go a bit faster thinking of it this way as you become more familiar with exponents.

$$\begin{array}{r|r}
 & 2x \\ \cline{2-2}
3x^2-2x+4 & 6x^3-7x^2+x-4
\end{array}$$

Step 3: Mulitply every term of the divisor by the first term of the quotient, and subtract the result from the dividend.

In this step we multiply the first term of the quotient $(2x)$ by each term in the divisor, $3x^2 - 2x + 4$, and write the result directly beneath the dividend.

$$\begin{array}{r|l}
 & \qquad\qquad\quad 2x \\ \cline{2-2}
3x^2-2x+4 & \;\;6x^3-7x^2+x-4 \\
 & -6x^3+4x^2-8x \\ \cline{2-2}
\end{array}$$

Immediately following, we subtract like terms and write those results down below the bar to give us a new dividend.

$$\begin{array}{r|l}
 & \qquad\qquad\quad 2x \\ \cline{2-2}
3x^2-2x+4 & \;\;6x^3-7x^2+x-4 \\
 & -6x^3+4x^2-8x \\ \cline{2-2}
 & \qquad -3x^2-7x-4
\end{array}$$

Step 4: Repeat steps two and three until the power of the divisor is less than the new dividend.

$$\begin{array}{r|l}
 & \qquad\qquad\quad 2x-1 \\ \cline{2-2}
3x^2-2x+4 & \;\;6x^3-7x^2+x-4 \\
 & -6x^3+4x^2-8x \\ \cline{2-2}
 & \qquad -3x^2-7x-4 \\
 & \qquad\quad 3x^2-2x+4 \\ \cline{2-2}
 & \qquad\qquad -9x
\end{array}$$

Step 5: Write the result in the correct form.

The result is is written as the following:

$$\text{quotient} + \frac{\text{last dividend}}{\text{divisor}} \quad \Rightarrow \quad 2x - 1 + \frac{-9x}{3x^2 - 2x + 4}$$

Example 1.8.5:

Divide $x^3 + x - 5$ by $x + 2$

Solution:

To begin, notice that the divisor and dividend are already in decreasing order, so all we have to do is perform steps 1 through 4 till we get to the end then write the result in the correct form.

$$\begin{array}{r|rrrr}
 & & x^2 - 2x & & +5 \\ \cline{2-5}
x+2 & x^3 & & +x & -5 \\
 & -x^3 & -2x^2 & & \\ \cline{2-3}
 & & -2x^2 & +x & \\
 & & 2x^2 & +4x & \\ \cline{3-4}
 & & & 5x & -5 \\
 & & & -5x & -10 \\ \cline{4-5}
 & & & & -15
\end{array}$$

Thus our answer is

$$x^2 - 2x + 5 + \frac{-15}{x+2}$$

Example 1.8.6:

Divide $x^4 + x - 5$ by $x^2 + 2$

Solution:

Again, both the dividend and divisor are already in decreasing order, so we just need to perform steps 1 through 4 repeatedly until the order of the new dividend is less than the order of the divisor.

$$\begin{array}{r|rrrr}
 & & x^2 & & -2 \\ \cline{2-5}
x^2+2 & x^4 & & +x & -5 \\
 & -x^4 & -2x^2 & & \\ \cline{2-3}
 & & -2x^2 & +x & -5 \\
 & & 2x^2 & & +4 \\ \cline{3-5}
 & & & x & -1
\end{array}$$

The result should be written as

$$x^2-2+\frac{x-1}{x^2+2}$$

All of the examples so far have been dividing a polynomial with another polynomial. In the case of dividing a multinomial by another multinomial where there are more than one variable, then the process is the same except we write the multinomial in decreasing order regardless of the variable.

Example 1.8.7:

Divide $x^3-xy^2+x^2y-y^3$ by $x+y$

Solution:

$$\begin{array}{r|l} & \quad\quad x^2 \quad\quad\quad -y^2 \\ \hline x+y & x^3+yx^2-y^2x-y^3 \\ & \underline{-x^3-yx^2} \\ & \quad\quad\quad\quad -y^2x-y^3 \\ & \quad\quad\quad\quad \underline{y^2x+y^3} \\ & \quad\quad\quad\quad\quad\quad\quad 0 \end{array}$$

Notice in this case that the remainder is zero, thus our answer is just x^2-y^2. However, since the remainder was zero then this means that the divisor was a factor of the dividend because it divided into it evenly with no remainder.

Chapter 2

Geometry

2.1 Lines, Angles, Triangles

It's important to understand the definitions in math, and in geometry there are a lot, but we use these terms later throughout this book extensively. Without the understanding of terms, then your understanding of other definitions that reference certain terms will be false.

Figure 2.1: Line AB

2.1.1 Lines, Rays, and Angles

A **line** is defined as the infinite straight line drawn between two points, including the points, and is denoted as AB or BA. It is common to not illustrate the point in the graph with a deliberate point such as •, unless emphasis needs to be put on it or a single point needs to be plotted. Figure 2.1 illustrates a line between points A and B.

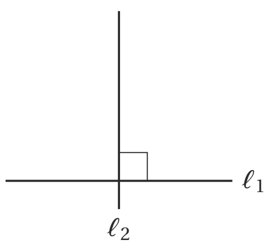
Figure 2.2: Ray

A **ray** (or **half line**) is a portion of a line drawn from a point, say A as shown in Figure 2.2, and passes through another point, say B. Since a ray will pass through other points, then any of the other points that the ray passes through can be used to denote the ray, thus the ray shown in figure 2.2 can be denoted as $\overrightarrow{AB}$.

Figure 2.3: Parallel Lines

When two lines, ℓ_1 and ℓ_2, are drawn so that they never cross then they are called **parallel** lines, and are denoted as $\ell_1 \mid\mid \ell_2$. When two lines intersect each other at a 90° angle then they are called **perpendicular** or **normal** . To denote whether two lines, again say ℓ_1 and ℓ_2, are perpendicular, we would write $\ell_1 \perp \ell_2$.

Figure 2.4: Perpendicular Lines

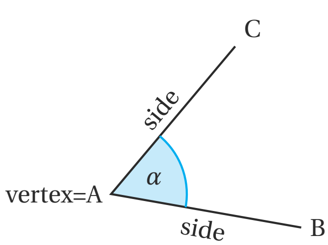


2.1.2 Angles

An **angle** is the amount of rotation of a ray about its vertex. Typically, an angle is identified with a lowercase Greek letter such as θ as shown in figure 2.5. The pivot point of rotation is called the **vertex** .

There are several ways to denote a particular angle within a figure if the angle isn't already identified. In figure 2.5 the angle is denoted as the lowercase Greek letter α; however, there are many instances where angles are not predefined this way. In figure 2.8b the angle at the vertex can be denoted as any one of the following: $\angle A$, $\angle CAB$, $\angle BAC$, or θ since it's the only angle in the figure . In this book, all angles will be referenced as either $\angle A$ (or any other capital Latin letter), or a lowercase Greek letter such as θ.

Figure 2.5: Angle



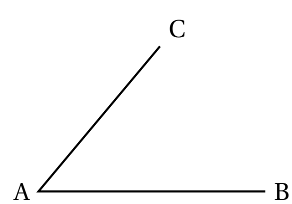
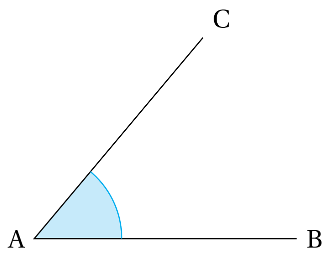
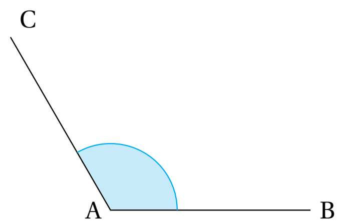


Figure 2.6

The following is a list of basic angles. A right angle, 90° angle, is always identified with small square at the vertex. Never assume that an angle is what it appears unless it is stated. For example, if an angle looks like a 90° angle but does not have the square to identify it as a right angle, then we can't assume that it is.

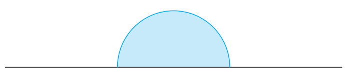


(a) Acute Angle

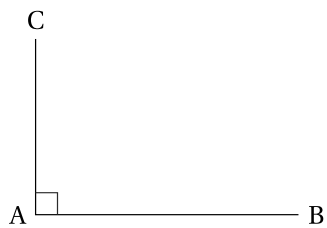


(b) Obtuse Angle

Figure 2.7



(a) Straight Angle



(b) Right Angle

Figure 2.8

Supplementary angles occur when the sum of two angles is equal to 180°. We would say that the supplement of 30° is 150° and vice versa. **Complementary angles** occur when the sum of two angles equal 90°. Thus, if we have an angle of 20°, then the complement of 20° is 70° and vice versa.

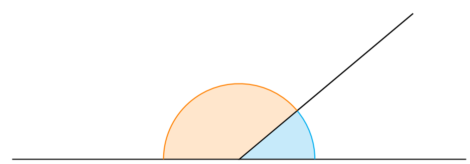


Figure 2.9: Supplementary Angles

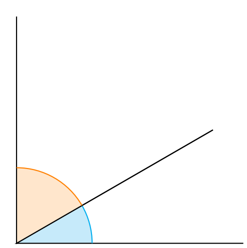


Figure 2.10: Complementary Angles

Example 2.1.1:

What are the complement and supplement of 40° ?

Solution:
The complement of 40°, for now we'll call it α , is the angle such that the sum of 40° and α equals 90°.

$$\begin{aligned} 40^\circ + \alpha &= 90^\circ \\ \alpha &= 90^\circ - 40^\circ \\ &= 50^\circ \end{aligned}$$

Thus, the complement of 40° is 50°.

The supplement of 40° is found similiarly except the sum of the two angles must equal 180°.

$$\begin{aligned} 40^\circ + \alpha &= 180^\circ \\ \alpha &= 180^\circ - 40^\circ \\ &= 140^\circ \end{aligned}$$

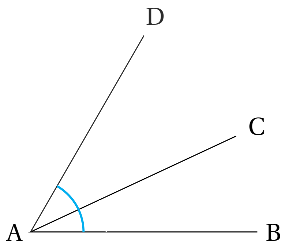


Figure 2.11: Adjacent angles

Two angles that share a vertex and one side are called **adjacent angles** as shown in figure 2.11. When two lines cross at a point, then they will create two sets of identical angles which are called **vertical angles** as shown in figure 2.12. A **transversal** is a line that intersects two lines; the lines can be either parallel or not. Figure 2.13 illustrates a transversal through two parallel lines. When two parallel lines are intersected by a transversal then the **corresponding angles** that are a result from the transversal create what is called **similar angles**. For example, $\angle A$ is similar or **congruent** to angles $\angle C$, $\angle E$, and $\angle G$. Likewise, $\angle B$ is congruent to $\angle D$, $\angle F$, and $\angle H$.

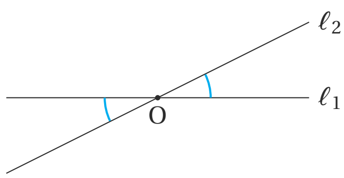


Figure 2.12: Vertical angles

In figure 2.15 all three lines denoted as ℓ_1, ℓ_2, and ℓ_3 are parallel, or symbolically $\ell_1 \parallel \ell_2 \parallel \ell_3$. When two or more parallel lines are intersected by two or more transversals then these intersecting lines create corresponding segments. **Corresponding segments** are the line segments created when two or more parallel lines are intersected by two or more transversals. Figure 2.14 highlights the line segment AB in blue. In figure 2.15 the line segment AB and DE are corresponding segments as is BC and EF; as is AC and DF. The ratios of corresponding segments can be used to find the length of an unknown segment since the ratios of corresponding segments are equivalent to each other. In other words:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

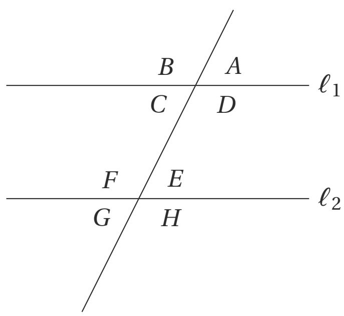


Figure 2.13

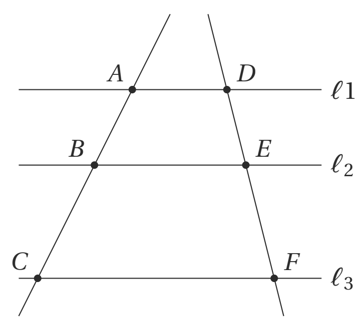


Figure 2.15: $\ell_1 \parallel \ell_2 \parallel \ell_3$

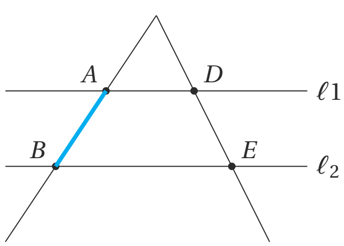
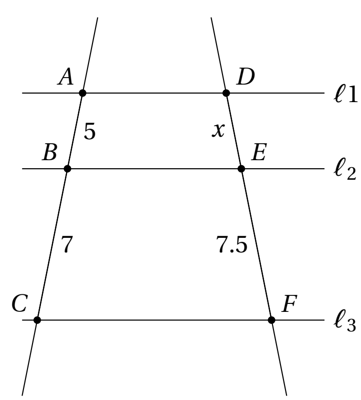


Figure 2.14: $\ell_1 \parallel \ell_2$

Example 2.1.2:

Find the line segment DE.



Though you can create an expression in several ways, it's best to set the problem up so that the unknown segment we are looking for, x, is in the numerator. This step is not necessary but it does simplify the solution.

$$\frac{x}{5} = \frac{7.5}{7}$$

$$= \frac{7.5(5)}{7}$$ multiply both sides of the equation by 5

$$= 5.3571$$

$$= 5.4$$ round result to least precise number

2.1.3 Degrees and Radians

Up till now we have seen angles measured in only degrees. While there are several units of measurement for an angle, in this book we will only look at two (degrees and radians). **Radians** is a unit of measurement that has a distinct relationship with the radius and the length of arc that the angle makes (more on this topic in chapter 3). For now, all we need to know is that there are 2π radians in a complete circle, and π radians in half a circle. In other words $180° = \pi$ radians. It's necessary to convert between degrees and radians often, thus we can use the relationship that $180° = \pi$ radians to develop a quick conversion between the two units.

To convert from degrees to radians, we need to know what 1 is in terms of the number of radians to degrees. Another way to think of this is that degrees is a unit of measurement, and when we multiply our given angle by our expression, the degree unit needs to cancel out.

Note: The steps taken to convert degrees to radian, or vice versa, are the same steps for all unit conversions.

$$180° = \pi \text{ rad}$$

$$\frac{180°}{180°} = \frac{\pi \text{ rad}}{180°}$$ divide both sides by 180°

$$1 = \frac{\pi \text{ rad}}{180°}$$

Thus, anytime we want to convert any angle thats given in degrees we just need to multiply it by $\frac{\pi \text{ rad}}{180°}$. However, when we refer to angles, it is not common to denote an angle in radians by writing "rad" appended to it, so from now on if an angle is not identified as degrees, then it is assumed to be in radians.

Example 2.1.3:

Convert 30° to radians.

Solution:
To convert 30° to radians, all we have to do is multiply our given degree measure by $\frac{\pi}{180°}$.

$$30° \cdot \frac{\pi}{180°} = \frac{30°\pi}{180°}$$
$$= \frac{\pi}{6}$$

Similarly, to convert from radians to degrees we can take the same relationship of $\pi = 180^\circ$ to determine the expression needed to multiply our given angle in radians by. We still need to find what 1 is in terms of the number of degrees to radians, thus we have:

$$\pi = 180^\circ$$
$$\frac{\pi}{\pi} = \frac{180^\circ}{\pi}$$
$$1 = \frac{180^\circ}{\pi}$$

Now, all we have to do to convert an angle given in radians to degrees is multiply it by $\dfrac{180^\circ}{\pi}$

Example 2.1.4:

Convert $\dfrac{5\pi}{6}$ to degree measure.

Solution:
To convert $\dfrac{5\pi}{6}$ to degrees we only need to multiply it by $\dfrac{180^\circ}{\pi}$.

$$\begin{aligned}\frac{5\pi}{6}\cdot\frac{180^\circ}{\pi} &= \frac{5\cdot 180^\circ\cdot\pi}{6\pi}\\ &= 150^\circ\end{aligned}$$

notice that π cancels out in the numerator and the denominator.

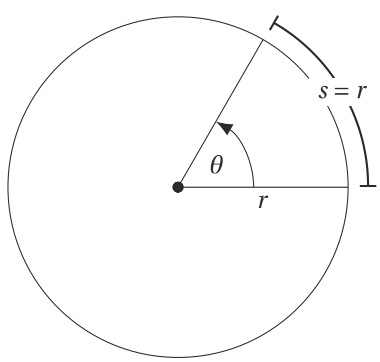


Figure 2.16: Degree and Radian

When the length of arc, denoted as s in figure 2.16, is equal to the radius, r, then this is defined as one radian.

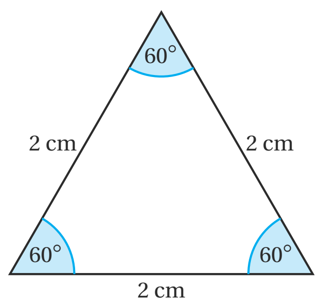
It is very important to understand the differences between degrees and radians. The radian unit of measurement refers to how many radii are along the circumference of an arc with respect to the radius; while degrees simply partitions a complete circle into 360 parts from a fixed point at the center. Thus, given a particular fixed angle and fixed radius, then the number of radii that are in the arc created by the angle and radius is also fixed which we call radians. Therefore, there is a direct correlation between the two units of measurement; however, their similarities end there. For now it is sufficient to know how to convert between them.

2.1.4 Polygons

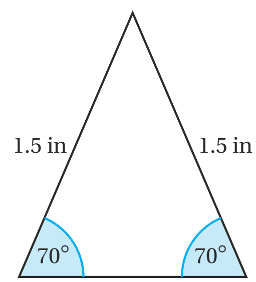
A **polygon** is a figure that is enclosed by three or more line segments. The number of sides a particular polygon has determines it's type. For instance a **triangle** has three sides, a **quadrilateral** has four sides, and **pentagon** has five sides. We look at polygons more closely in section 2.2

2.1.5 Triangles

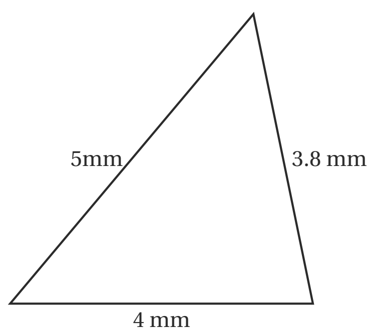
A **triangle** is a polygon that has three sides. The type of triangle is determined by the properties of the sides, and/or angles. An **equilateral triangle** has all sides equal in length, thus all angles are equal as well. An **isosceles triangle** has two sides of equal length, thus each adjacent angle is equal as well. The **scalene triangle** has no two sides the same. Perhaps the most important triangle that we'll be looking at is the **right triangle** where one angle within the triangle is $90°$. The $90°$ angle in a right triangle is always opposite the hypotenuse. Right triangles are used extensively in trigonometry.



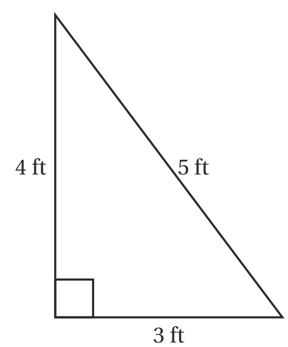
(a) Equilateral triangle



(b) Isosceles triangle



(c) Scalene triangle



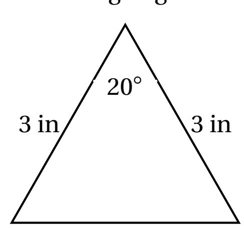
(d) Right triangle

Figure 2.17

The sum of the angles in any triangle is always $180°$. Because of this property, if we are either given, or can determine, any two angles of a triangle, then the last angle is easily found by subtracting the sum of the two known angles from $180°$. Likewise, using properties of the different types of triangles, we may be able to find two missing angles given only one angle.

Example 2.1.5:

Find the measure of the two missing angles in degrees.



Solution:

Since both sides are the same length then we know this is an isosceles triangle where the two adjacent angles to those sides are also the same. In addition, since all angles in any triangle sum up to 180° the remaining two angles, lets denote them each as x, can be found by taking the difference of 180° and 20° and dividing by two.

$$\begin{aligned} 20^\circ + 2x &= 180^\circ \\ 2x &= 180^\circ - 20^\circ \\ 2x &= 160^\circ \\ x &= \frac{160^\circ}{2} \\ x &= 80^\circ \end{aligned}$$

The two missing angles are 80°

The **perimeter** of a triangle is the distance around the triangle and can be found by summing up all sides. For example, in figure 2.18 the length of the sides are denoted generically as sides a, b, and c. To find the perimeter we just need to add them up.

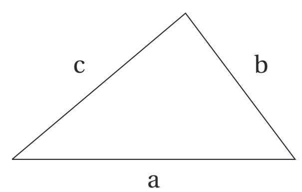


Figure 2.18: Triangle

$$\text{perimeter(P)} = a + b + c$$

Example 2.1.6:

A triangular piece of scrap metal has sides that measure 15.6cm, 9.07cm, and 19.33cm. What is the perimeter so the piece of scrap metal?

Solution:

$$\begin{aligned} P &= a + b + c \\ &= 15.6 + 19.33 + 9.07 \\ &= 44.0cm \end{aligned}$$

recall that your answer can only be as precise as your least precise number. Thus, the answer must be precise to the tenths place, so the zero is necessary.

A segment drawn from a vertex of a triangle to the midpoint of the line opposite the vertex is called a **median** . The intersection of all three medians of any triangle denotes the center of gravity which we call the **centroid** . The universal symbol ◕ is used to denote the centroid; however, we will simply use C in this text. Figure 2.19 illustrates the centroid of a triangle found by the intersection of all three medians.

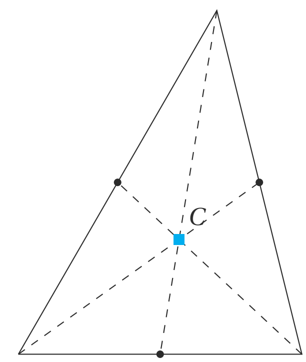


Figure 2.19

The **area of a triangle** can be found in several ways. First, if the height, or **altitude** , and the width of the base of a triangle is given or determined, the area can be found by the product of 1/2 times the base times the altitude.

$$\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$$

or

$$A = \frac{1}{2} bh$$

Example 2.1.7:

Find the area of a triangle whose base is 3 feet, and altitude of 8 feet.

Solution:

$$\begin{aligned} A &= \frac{1}{2} bh \\ &= \frac{1}{2} \cdot 3 \cdot 8 \\ &= \frac{1}{2} \cdot 24 \\ &= 12\text{ft}^2 \end{aligned}$$

notice the unit is now ft^2
which denotes square feet

Another method for determining the area of a triangle is called **Heron's formula**, or **Hero's formula** . Hero's formula is useful when the height of a triangle is unknown. For example, if we had a triangle that isn't a right triangle and wasn't given the altitude such as the triangle illustrated in figure 2.20 , we would find it difficult to determine mathematically what the altitude was; however, if we know the lengths of all the sides then we can still determine the area using Hero's formula.

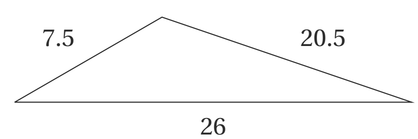


Figure 2.20: Triangle

Hero's formula is defined as follows: we define a variable s such that s is the perimeter of the triangle divided by two or

$$s = \frac{P}{2} = \frac{a+b+c}{2}$$

Then the area of the triangle is given by,

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Example 2.1.8:

Find the area of the triangle in figure 2.20.

Solution:

First we must determine s by taking the perimeter and dividing it by two.

$$\begin{aligned} s &= \frac{7.5+20.5+26}{2} \\ &= \frac{54}{2} \\ &= 27 \end{aligned}$$

Now, we replace s with 27 in our formula. Keep in mind that it does not matter which sides you've chosen for a, b, or c as they are arbitrary.

$$\begin{aligned} A &= \sqrt{27(27-7.5)(27-20.5)(27-26)} \\ &= \sqrt{27(19.5)(6.5)(1)} \\ &= \sqrt{3422.25} \\ &= 58.5 \end{aligned}$$

the original values did not indicate the units; however, the answer is in square units.

A common triangle used in construction is called the 3,4,5 triangle. The main property of this triangle is that it is a right triangle. In construction if you measure two walls out from a single pivot of three feet and four feet then the hypotenuse should measure 5 feet if the two walls do in fact form a 90○ angle to one another. This also works for any multiple of the 3,4,5 triangle such as 30', 40', and 50'. However, there is a better method to accomplish this task that will allow us to use the exact measurements of the walls even if they're not a multiple of the 3,4,5 triangle and it is called the **Pythagorean Theorem**.

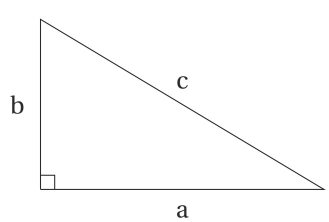


Figure 2.21: Right Triangle

Pythagorean Theorem

$\triangle abc$ as shown in figure 2.21 is a right triangle if the sum of the squares of the two legs, a and b is equal to the hypotenuse, c, squared.

$$a^2 + b^2 = c^2$$

Example 2.1.9:

A communication tower shown in figure 2.22 is 100 meters tall. A cable is anchored at the top of the tower and 20 meters from the base of the tower. How long is the cable?

Solution:

The tower and cable can be illustrated by using a right triangle such as the one shown below. Since this is a right triangle, we can use the Pythagorean theorem to calculate the length of the cable which is denoted as x in Figure 2.22. It does not matter which legs of the right triangle you reference as a, or b; however, side c must always be the side opposite the 90° angle.

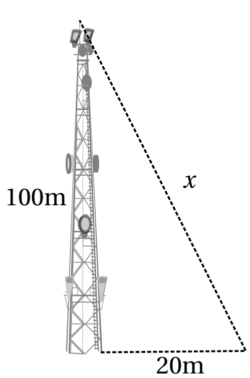
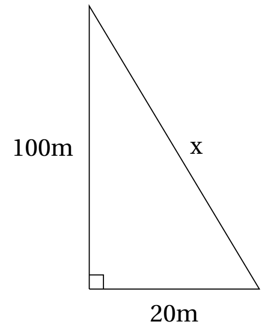


Figure 2.22: Communication Tower



Now, we have the following equation.

$$x^2 = 20^2 + 100^2$$

$$x = \sqrt{20^2 + 100^2}$$

$$x = 102m$$

Recall the answer must be rounded to the least precise number

Notice that there was no intermediate calculations such as squaring the 20 and 100 then taking the square root. The less intermediate calculations we do, the less intermediate rounding we have to do. This will help ensure our answers are more accurate. Todays calculators can perform many calculations in one step.

Example 2.1.10:

If one leg of a right triangle is 3.25 feet long, and the hypotenuse is 5.5 feet long, what is the length of the second leg of the triangle?

Solution:

Although it is helpful to draw a diagram of the problem, it's sometimes not necessary such as in this instance. Thus, we begin with the Pythagorean theorem and substitute what we know.

$$a^2 + b^2 = c^2$$

$$a^2 + 3.25^2 = 5.5^2$$

we could have chosen to substitute 3.25 for a

$$a^2 = 5.5^2 - 3.25^2$$

$$a = \sqrt{5.5^2 - 3.25^2}$$

$$a \approx 4.4ft$$

2.2 Polygons

Figure 2.23: Hexagonal Bolts

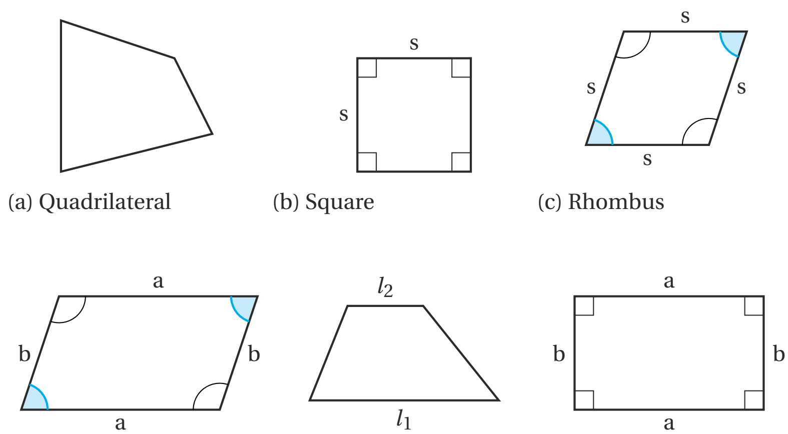
The hexagon is the most used shape for nuts and bolts mainly because it provides plenty of surface area to grip by different tools without excessive stress on both the tool and bolt head.

In this section we look at polygons with at least four sides. In section 2.1 we looked at the simplest form of a polygon which is the triangle. In this section we will briefly introduce a handful of polygons and some of their properties.

After the triangle the next simplest polygon is called a **quadrilateral** meaning shapes of exactly four sides. A **pentagon** is a polygon with five sides, and a **hexagon** has six sides. All of these shapes have similar and sometimes different properties which makes each of them very useful in certain applications.

2.2.1 Quadrilaterals

A quadrilateral is any four sided shape. Similar to that of triangles, quadrilaterals have different names when they exhibit different properties. For example, a **square** is a quadrilateral that has four equal sides with all angles measuring 90°. A **parallelogram** has two pairs of sides opposite each other that are parallel. Consequently, each pair of opposite sides to each other are also congruent. Figure 2.25 shows a set of special quadrilaterals where each has a set of different properties aside from simply being four sided figures.



(a) Quadrilateral (b) Square (c) Rhombus

(d) Parallelogram (e) Trapezoid ($l_1 || l_2$) (f) Rectangle

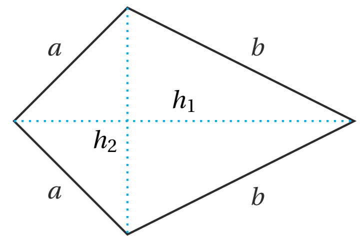
Figure 2.24

While there are other special quadrilaterals with specific properties other than those shown in figure 2.25 such as the **kite** where the kite has two pairs of congruent adjacent sides, we will concentrate on the six quadrilaterals illustrated in figure 2.25.

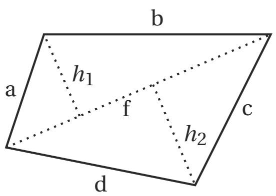
A **trapezoid** has two sides, say l_1 and l_2, such that l_1 and l_2 are parallel; symbolically we say that $l_1 || l_2$. A **rectangle** is similar to a square in that all adjacent sides form a 90° angle to one another; however, unlike the square where all sides are congruent, the rectangle has opposite congruent sides. Again as a consequence, the pairs of opposite congruent sides also are parallel. Also, notice by these definitions that some quadrilaterals such as the trapezoid may fit the definition of other quadrilaterals. For example, since a rectangle has two pairs of opposite parallel sides, then it technically fits the definition of a trapezoid, but the converse of this statement is not true. This means that properties of the trapezoid such as the equation for area would also apply to that of the rectangle. This becomes very useful when applying properties of quadrilaterals to approximate the area of irregular shapes. Lastly, the **Rhombus** is similar to that of the parallelogram where the rhombus has two pairs of parallel sides; however, the sides of the rhombus are all the same length.

The **perimeter** of any polygon is the sum of all of the sides. Some shapes are made easier to calculate the perimeter based upon their properties. For example, since the square has all sides of equal length, then the perimeter would simply be $4s$ where s is the length of one of the sides. To find the perimeter of any quadrilateral we just have to sum up all four sides.

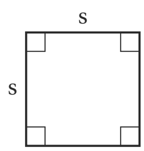
To calculate the **area of a quadrilateral**, we typically have to take the length of at least one of the sides and the distance between the sides. In the case of the square, to determine the area we take the length of the side, say s, and multiply it by the distance between the sides which is also s, thus we have area $= s \cdot s = s^2$. The following are the formulas for calculating the perimeter and area of some polygons.



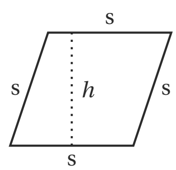
A kite is a quadrilateral that has two adjacent pairs of sides that are congruent. The perimeter of the kite can be determined by summing up the lengths of the sides, thus Perimeter$(P) = 2a + 2b$. The area of the kite can be determined by Area$(A) = \frac{1}{2} h_1 h_2$



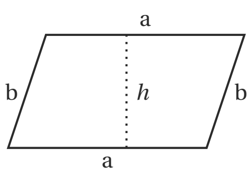
(a) Quadrilateral
P=a+b+c+d
A=$\frac{1}{2} \cdot f \cdot (h_1 + h_2)$



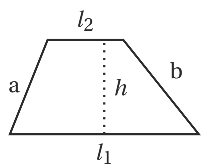
(b) Square
P=$4 \cdot s$
A=s^2



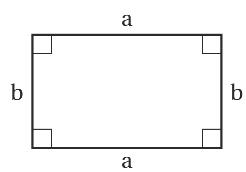
(c) Rhombus
P=$4 \cdot s$
A=$h \cdot s$



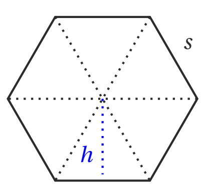
(d) Parallelogram
$P = 2a + 2b$
$A = h \cdot a$



(e) Trapezoid
$P = l_1 + l_2 + a + b$
$A = \frac{1}{2} h(l_1 + l_2)$



(f) Rectangle
$P = 2a + 2b$
$A = a \cdot b$



(g) Hexagon
$P = 6s$
$A = 3sh$
$= \frac{3\sqrt{3}}{2} s^2$
$= 2\sqrt{3} h^2$

Figure 2.25

Example 2.2.1:

Find the perimeter of a parallelogram with sides that measure 3.25 cm, 5.125 cm, 3.25 cm, and 5.125 cm.

Solution:

To determine the perimeter we need to sum up the sides while keeping in mind that the result can only be as precise as our lest precise number since these are measurements. If we knew that the values were exact then we could leave the answer correct to however many decimal values our calcualtor gave.

$$\begin{aligned} P &= 2a + 2b \\ &= 2(3.25) + 2(5.125) \\ &= 16.75\ cm \end{aligned}$$

Example 2.2.2:

Find the area of a trapezoid with bases that measure 20 m, and 30 m, while the distance between the two bases (h) measures 10 m.

Solution:

$$\begin{aligned} A &= \frac{1}{2}h(l_1 + l_2) \\ &= \frac{1}{2}(10)(20+30) \\ &= \frac{500}{2} \\ &= 250\ m^2 \end{aligned}$$

notice that final answer is in square meters

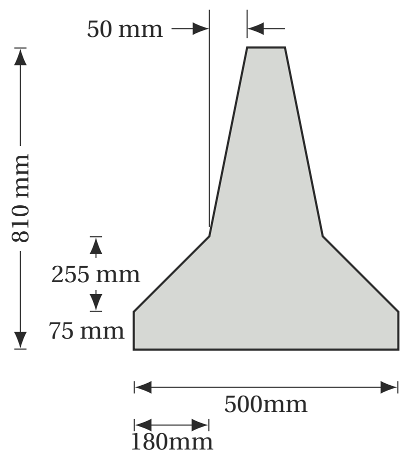


Figure 2.26: Concrete Barrier

Example 2.2.3:

Figure 2.26 illustrates the cross section of a concrete barrier. Determine the area of the cross section in square meters.

Solution:

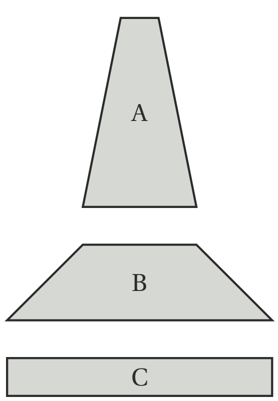


Figure 2.27: Barrier Components

The illustration, as it stand, does not represent any of the polygons that we've seen thus far; however, we can break up the cross section into several components as shown in figure 2.27, calculate the areas of each, and then add them up to get the cross sectional area of the entire barrier. All measurements are in millimeters, so we will have to convert the units to meters at the end.

For section A we need to determine the lengths of the two bases. Neither is explicitly given, but enough information is shown to determine the lengths. The first base of A (top of A) is $l_1 = 500 - 2(180) - 2(50) = 40mm$. l_2 can found with the same method: $l_2 = 500 - 2(180) = 140mm$. We also need the height of section A which is $h = 810 - 255 - 75 = 480mm$, thus the area for A is

$$\begin{aligned}\text{Area}_A &= \frac{1}{2}h(l_1 + l_2)\\ &= \frac{1}{2}(480)(40+140)\\ &= 43200\ mm^2\end{aligned}$$

Section B is also a trapezoid where the top length is the same as the bottom length of section A. The bottom and height is given, thus we have

$$\begin{aligned}\text{Area}_B &= \frac{1}{2}h(l_1 + l_2)\\ &= \frac{1}{2}(255)(140+500)\\ &= 81600\ mm^2\end{aligned}$$

Section C is a rectangle so its area is

$$\begin{aligned}\text{Area}_C &= ab\\ &= 75(500)\\ &= 37500\ mm^2\end{aligned}$$

The total area, T, in square millimeters is sum of all the sections:

$$\begin{aligned}\text{Area}_T &= \text{Area}_A + \text{Area}_B + \text{Area}_C\\ &= 43200 + 81600 + 37500\\ &= 162300\ mm^2\end{aligned}$$

However, we were asked to find the area of the barrier cross section in square meters. Since there are 1,000,000 square millimeters in one square meter we have

$$\begin{aligned}m^2 &= \frac{mm^2}{1,000,000}\\ &= \frac{162300\ \cancel{mm^2}}{1,000,000\ \cancel{mm^2}}\\ &= 0.1623\ m^2\end{aligned}$$

Note:
$1mm = .001m$, thus there are $1000mm$ in 1 meter.

$$1000mm \times 1000mm = 1,000,000mm^2$$

2.3 Circles

Unlike polygons where a shape is defined by segments of straight lines, Circles have no straight lines; however, components of a circle do. In this section we look at the circle and its parts. As with every shape we've seen so far, the names and definitions of the components of a shape are incredibly important. For example, If I gave you the equation for pi, $\pi = \frac{C}{d}$, you could also represent this same ratio using the radius instead of the diameter of a circle. Ultimately, it is up to you to recognize what components, or parts, are given, and whether or not a substitution is needed.

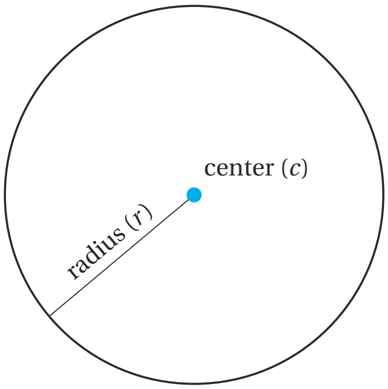


Figure 2.28

2.3.1 Components of a Circle

The circle has many parts where each part has specific properties. The **center**, denoted by c, is probably the most recognized part and the one part that is most self explanatory. Nevertheless, the center of a circle is located at a point that is equidistant to every point along the circle. The **radius** is the distance from the center to any point on the circle, and is denoted by r. The radius and the center is shown in figure 2.28. A **chord** is a line segment that touches at two locations of the circle. The **diameter** is a line segment that passes through the center, and touches the circle at two locations. Consequently, the diameter is also a chord. The **tangent** line intersects the circle at only one location, or point, thus does not pass through the circle. A **secant** passes throught the circle while intersecting the circle in two locations. The **circumference** is the perimeter of the circle. Figure 2.29 illustrates most parts of the circle defined here.

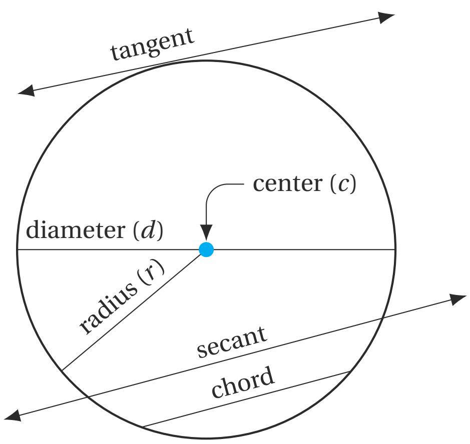


Figure 2.29

One property with regard to the relationship of a tangent line and the radius of a circle that's certainly worth mentioning is the tangent line (any) creates a 90° angle with the radius at the point of intersection as shown in figure 2.30.

There are other names for parts of a circle. For example, if we partitioned the circumference of a circle into two parts, then the result will give us a major and minor arc. The section, or region, inside the circle as a result of the partition is called a **sector** as can be seen in figure 2.31.

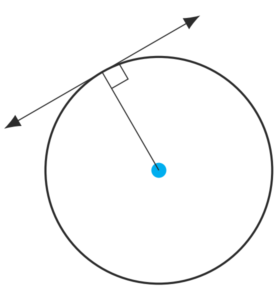


Figure 2.30

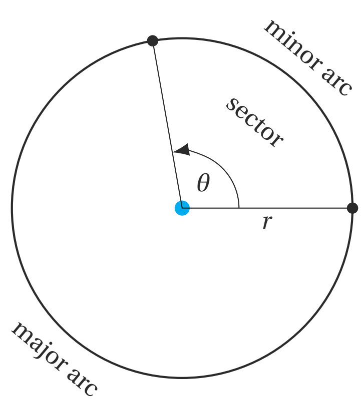


Figure 2.31

2.3.2 Circumference and Area of a Circle

As stated earlier in this section, the perimeter of a circle is called the circumference. The equations for both the circumference and the area both involve the irrational number $\pi \approx 3.1415$. While it is sometimes preferable to round π to a certain precision, our calculations will be more precise while using the physical key, $\boxed{\pi}$, on the calculator. In this text, the physical π key will always be used unless otherwise stated or instructed. Below are equations for the circumference and area of a circle.

Circumference and Area of a Circle

$C = \pi d = 2\pi r$	Circumference of a circle with radius r, or with diamter d.
$A = \pi r^2 = \dfrac{\pi}{4} d^2$	Area of a circle with radius r, or with diameter d.

Example 2.3.1:

Find the area and circumference of a circle given that the circle has a diameter of 203.2mm.

Solution:

While it is usually a preference, some students prefer to use the radius instead of the diamter. This could be due to forgetting the equations with respect to the diameter, or simply prefer the equation with the radius. Nevertheless, since we have equations with respect to the diameter, then we'll use those equations.

$$C = \pi d = \pi(203.2mm) = 638.4mm$$

recall that we must round the result to the least precise number. In addition, we use the $\boxed{\pi}$ key on our calculator. Refer back to section 1.4 to see why it's not correct to round π to one decimal place also.

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4}(203.2)^2 = 32429.3mm^2$$

note that it's not necessary to put parentheses around 203.2 at all, rather the parentheses was used here to imply multiplication and to keep things legible.

Figure 2.32 shows the calculator output.

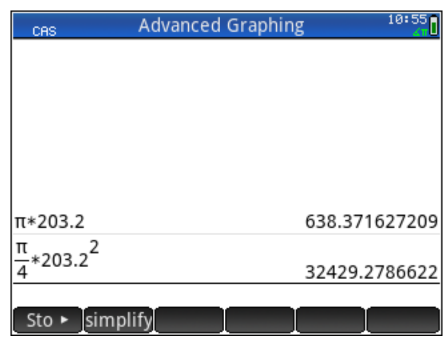


Figure 2.32

Example 2.3.2:

What radius and circumference is required if the area of a circle is to have 78.5 square inches?

Solution:

Since we are given the area of the circle, then we begin by finding the radius using the equation for area.

$$\begin{aligned} A &= \pi r^2 \\ 78.5 &= \pi r^2 \\ \frac{78.5}{\pi} &= r^2 \\ \sqrt{\frac{78.5}{\pi}} &= r \\ r &= 4.99873in \\ &\approx 5in \end{aligned}$$

rounding to the tenths place here forces us to round to 5

Thus, the radius will be 5 inches. From this we can determine the Circumference of the circle by using the equation for circumference.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi(5) \\ &= 31.4in \end{aligned}$$

The radius and circumference of a circle with an area of 78.5 in^2 is 5 in, and 31.4 in respectfully.

2.3.3 Sectors and Arcs

An arc is created by partitioning the circumference of a circle into two parts. This partition creates a **central angle**, θ, with respect to the center of the circle as shown in figure 2.33. When referencing an arc such as the minor arc in figure 2.33, we write $\overset{\frown}{AB}$.

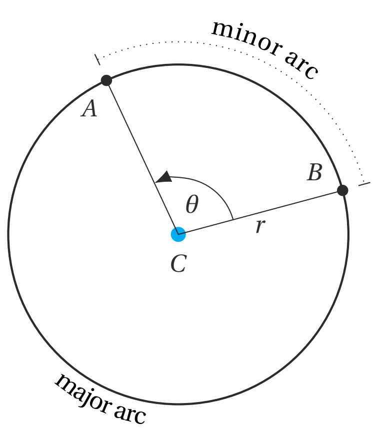


Figure 2.33

Arclength

The length of an arc, denoted s, created from a circle of radius r and angle θ in (radians) is defined by:

$$s = r\theta$$

Now that we have the formal definition of the arc length, it's important to understand that the arc length, $s = r\theta$, is the *linear* length of the arc. Moreover, this is also why when determining the length of arc that the angle θ must be in radians. Figure 2.34 illustrates the distance a circle, or wheel, has traveled with a certain degree measure and radius. The distance the wheel has traveled is the same length as the arc length. See sub-section 2.1.2 on page 51 to review the relationship between degrees and radians if needed.

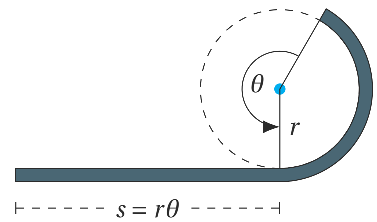


Figure 2.34

Example 2.3.3:

Find the arc length of a circle defined by a central angle of 60° and a radius of 9 inches.

Solution:

First the equation for arc length requires that the central angle, θ, must be in radians

$$\theta = 60^\circ * \frac{\pi}{180^\circ}$$
$$= \frac{\pi}{3}$$

recall that angles that aren't defined in degrees are assumed to be in radian, thus it's not necessary to write $\frac{\pi}{3}\ rad$.

Now that we have the central angle in radians we just need to multiply it by the radius to find the arc length

$$s = r\theta$$
$$= 9\left(\frac{\pi}{3}\right)$$
$$= 3\pi$$

since π represents the irrational number 3.1415..., then leaving the answer in this form is useless to us unless precision is needed later on.

$$\approx 9.424778$$

Since the radius and central angle was given and not specifically stated that they were measured, we have to assume the two were exact values, thus we don't round the final answer to 9 inches. Note: in instances such as this, rounding to a certain decimal value will often be asked such as in homework problems.

The next example is a direct application of arc length to determine the number of rotations a bicycle wheel makes with every turn of the crank. This example illustrates that the arc length is not bound to just a section, or part, of the circumference, rather the arc length can be the result of many revolutions of a circle such as a wheel, gears, pulleys etc.

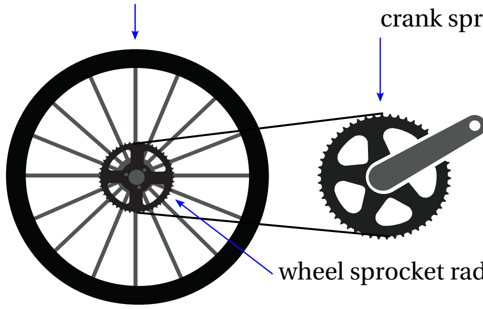


Figure 2.35: Bicycle Crank Components

Example 2.3.4:

Figure 2.35 above illustrates the propulsion mechanism for a bicycle where the wheel has a radius of 15.5 in., the small sprocket attached at the center of the wheel has a radius of 2 in., and the crank sprocket has a radius of 4.75 in. If the crank/pedal is rotated 180°, how far does the bicycle travel? Round the final result to the nearest hundredth.

Solution:

To begin, we need to find the arc length (i.e linear distance) of the crank sprocket that has a radius of 4.75". To do this, we first need to convert 180° to radians which would be $\theta = 180^\circ \dfrac{\pi}{180^\circ} = \pi$

$$\begin{aligned} s &= r\theta \\ &= 4.75\pi && \text{substitute } r = 4.75 \text{ and } \theta = \pi \end{aligned}$$

Thus, the chain moves approximately 14.9 inches; however, we want to avoid intermediate rounding so we use 4.75π and round the final answer.

To find the angle of the wheel sprocket we use the same equation as above and solve for θ. Keep in mind that the variables in the equation now refer to the wheel sprocket.

$$\begin{aligned} r\theta &= s \\ 2\theta &= 4.75\pi && \text{substitute } s \text{ for the arc length} \\ & && \text{we just previously found} \\ \theta &= \frac{4.75\pi}{2} && \text{recall } \theta \text{ here is the angle for} \\ & && \text{the wheel sprocket.} \\ \theta &= 2.375\pi && \text{2.375 is still exact. no} \\ & && \text{rounding took place.} \end{aligned}$$

So, the wheel sprocket, and consequently the wheel, has rotated by an angle of approximately 7.46 radians. Therefore, the wheel will travel a distance of:

$$\begin{aligned} s &= 15.5(2.375\pi) \\ &\approx 115.65in \end{aligned}$$

The bicycle traveled 115.65 inches.

Recall that the definition of a sector found in subsection 2.3.1 on page 65 is part of a circle formed by an angle. To determine the area of a sector such as the shaded region shown in figure 2.36, we will need the same components of the sector as that of determining the arc length.

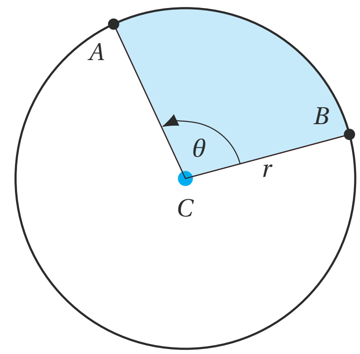


Figure 2.36

Area of a Sector

The area, A, formed by a circular arc of radius, r, and angle θ (in radians) is defined by the equation:

$$A = \frac{1}{2}r^2\theta$$

Example 2.3.5:

Determine the area of a sector formed by an angle of 30° and a radius of 5 cm. Round result to three decimal places.

Solution:

Since the equation for the area of a sector requires our angle to be in radians we have $\theta = 30° = \frac{\pi}{6}$.

$$\begin{aligned} A &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}(5^2)\frac{\pi}{6} \\ &= \frac{25\pi}{12} \\ &\approx 6.545 \text{ cm}^2 \end{aligned}$$

The area of the sector is approximately 6.545 cm^2.

2.4 Geometric Solids

So far all the figures that we've seen are plane figures, meaning they are are two dimensional figures. Two dimensional figures are still important when understanding how to calculate the volume of a solid geometric shape. **Volume** is the number of cubic units that occupy the entire space of a particular 3-dimensional shape. Some of the objects we look at are fairly intuitive when it comes to understanding the equations for for calculating volume such as cylinders. Typically, many people think of a cylinder as just a round tube, hollow or not, but this is not completely correct.

When referring to cylindrical or circular objects, the term *elements* comes up often. An **element**, in geometry, is a term used to describe a line segment, or set of line segments that form the side of a circular solid. For example, the line labeled h for the *right circular cylinder* in figure 2.37 below is an *element*. The set of all elements of form a cylinder.

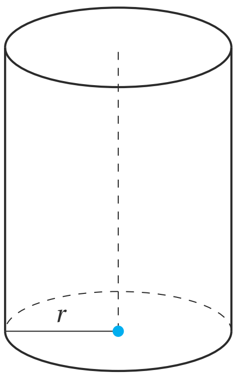


Figure 2.37: Right Circular Cylinder

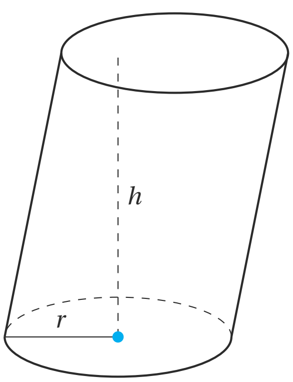


Figure 2.38: Oblique Cylinder

2.4.1 Surface Area

There are two types of surface areas described in this section which is *total surface area*, and *lateral surface area*. The **lateral surface area** is the area of all the sides combined, but not including the top and bottom sections. The **total surface area** is the area is the sum of the lateral surface area, the area of the top, and the are of the bottom. Surface areas are always measured in square units. For example, if we wanted to know how much much paint it would require to paint a circular silo (storage container for grain on a farm), then we would need to know the lateral surface area which excludes the dome shape at the top. If we wanted to know how much grain the silo could hold, then we would want to know the volume of the silo.

2.4.2 Cylinders

A **Cylinder** is any 3-dimensional shape where the top and bottom are both circular and congruent, and any corresponding opposite elements are parallel. For example, a rod where the top and bottom faces are congruent, but have an elliptical shape would still be considered a cylinder. In this book, we look at two types of cylinders which are called the circular cylinder, and the right circular cylinder. A **right circular cylinder** is formed when the elements are perpendicular to the base. The **oblique cylinder** is formed when the elements of the cylinder are not perpendicular to the base. The equations for calculating volume, and surface areas are the same. The **height**, also called the **altitude**, is the distance measured *perpendicular* to the top and bottom faces of the cylinder.

Below are the equations for determining the volume and surface areas of a cylinder. However, it's important to understand that when any shape has sides that are parallel to each other, then the volume can be calculated by finding the area of the cross-section of the shape and multiply it by its length. For example, we now know that the are for a circle is $A = \pi r^2$, so to determine the volume of a cylinder we just muliply the area by it length, or height.

Volume and Surface Area of a Cylinder

The volume, V, lateral surface area, L, and total surface area, T of a cylinder with radius r, and altitude h is:

Figure:	Volume:	Lateral Area:	Total Area:
Cylinder	V=$\pi r^2 h$	$L = 2\pi r h$	$T = 2\pi r h + 2\pi r^2$

2.4.3 Prisms

A **prism** is a solid geometric figure where the two faces (top and bottom) are congruent, and whose sides are parallelograms that are also parallel to all other adjacent or opposite sides. The height, h is the distance between the two faces. Since the cross section of a prism can be one of many different shapes, the equation for determining the volume will be different; however, the volume will can be determined in the same manner as that of a cylinder. To calculate the volume of a prism, first determine the area of the cross section (either the top or bottom), then multiply it by the height.

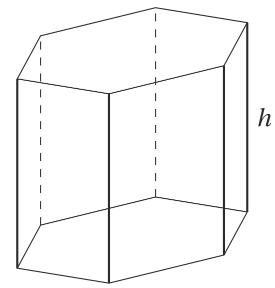


Figure 2.39: Hexagonal Prism

A prism's name is determined by the polygon that makes up its bases. If a prism has a triangle for a base, then it's called a **triangular prism**. Moreover, if a prism has a hexagon for a base then it's called a **hexagonal prism** as can be seen in figure 2.39. While a prism with a rectangular base and perpendicular sides is appropriately called a **rectangular prism**, it is also known as **rectangular parallelpiped**. In addition, a **cube** is also a prism and can also be referred to as a**right square prism**.

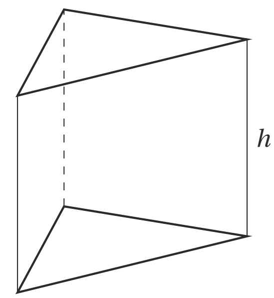


Figure 2.40: Triangular Prism

Determining the volume of a rectangular, or square, prism is probably the simplest volume calculation we'll have. Since the area of the base is determined by multiplying the length and width, and in the case of the square we have $w \cdot w = w^2$, then to find volume we just multiply the area of the base times the height.

$$V_{\text{rectangular prism}} = lwh$$

In the case of a cube. we have:

$$V_{\text{cube}} = w^3$$

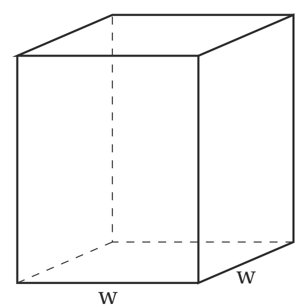


Figure 2.41: Rectangular Parallelpiped Prism

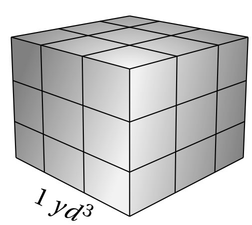
Example 2.4.1:

How many cubic feet are in one cubic yard?

Solution:

One cubic foot is one foot cubed or $(1\ ft)^3 = 1^3\ ft^3 = 1\ ft^3$. Since there are 3 feet in one yard, then we can calculate how many cubic feet are in one cubic yard in the same manner by substituting 1 yard with 3 feet.

$$\begin{aligned} V &= w^3 \\ (1\text{ yd})^3 &= (3\ ft)^3 \\ &= 3^3\ ft^3 \\ &= 27\ ft^3 \end{aligned}$$



There are 27 cubic feet in one cubic yard.

Example 2.4.2:

A building is built with a base measuring 32 feet by 40 feet. A slab of concrete is to be poured by the same dimensions and 4 inches thick. How many cubic yards of concrete must be ordered if the concrete supplier will deliver in half-cubic yard increments up to 16 cubic yards?

Solution:

First, we have to make a decision. Do we want to perform our calculation in feet or yards? Since we have most our measurements in feet already, we can proceed in feet (it doesn't matter which units we use). Now, that we've chosen the units we'll work with, all our measurements must be in the same units. Since the depth is give in inches we need to convert it to feet, thus we have

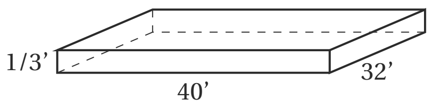


Figure 2.42

$$\begin{aligned} 4" &= \frac{4\ in}{1} \cdot \frac{1\ ft}{12\ in} \\ &= \frac{1}{3}\ ft \end{aligned}$$

Second, it's always a good idea to draw a picture of the problem as shown in figure 2.42. In this instance, the problem is not so complex that an image can't be pictured in our minds eye; however, drawing a diagram always helps. The volume of the slab, in cubic feet, is

$$\begin{aligned} V &= \frac{1}{3}(32)(40) \\ &= \frac{1280}{3} \\ &\approx 426.67\ ft^3 \end{aligned}$$

rounding to hundredths, or tenths, here is precise enough since we have to deal in half cubic yards anyway.

The slab contains approximately 426.67 cubic feet, but we need the answer in cubic yards. Since we know from the previous example that there are 27 cubic feet in one cubic yard, we have $426.67/27 \approx 15.8\ \text{yd}^3$. There are 15.8 cubic yards of concrete needed to fill the slab, The minimum number of cubic yards needed to be ordered is 16.

2.4.4 sphere

A **sphere** is a round object such that all points are equidistant from its center. The **radius** of a sphere is the distance from the center to any point on the surface of the sphere. The **diameter** is the distance between two points on the surface of the sphere that also passes through the center.

Volume and Surface Area of a Sphere

The volume, V, and total surface area, T of a sphere with radius r is:

Figure:	Volume:	Lateral Area:	Total Area:
Sphere	$V = \frac{4}{3}\pi r^3$	N/A	$T = 4\pi r^2$

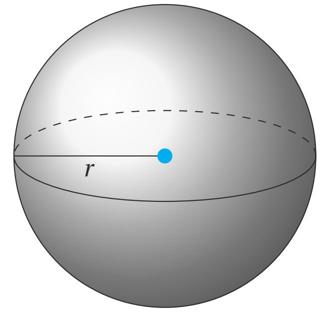


Figure 2.43: Sphere

2.4.5 Circular Cones

A **right circular cone** is a shape formed from a circular base which tapers to a point called the **vertex** that is perpendicular to the center of the base. Consequently all *elements* of the right circular cone are congruent. The *right circular cone* is the most frequently used form of cone that is used; for this reason it will often be referred to as a **cone**. Similiarly to the equations for cylinders, the equations for both volume, and surface areas are the same for both types of cones (right circular cone, and oblique cone).

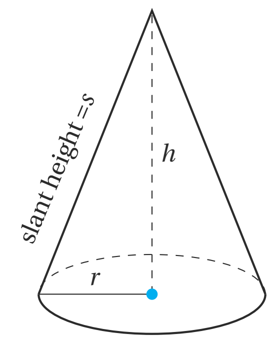


Figure 2.44: Right Circular Cone

Volume and Surface Areas of Cones

The volume, V, lateral surface area, L, and total surface area, T of a cone is given by:

Figure:	Volume:	Lateral Area:	Total Area:
Cone	$V = \frac{1}{3}\pi r^2 h$	$L = \pi r s$	$T = \pi r(r+s)$

With circular base radius, r, altitude, h, and slant height s.

note: Since the slant height for an *oblique* cone is undefined, there is no equation to determine the lateral surface area.

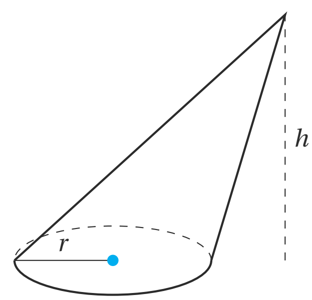


Figure 2.45: Oblique Cone

2.4.6 Pyramid

A **pyramid** is any object where the base is a polynomial, and the sides converge to a point called the **vertex**. Sides of a pyramid form a triangle, and are referred to as **lateral faces**. When determining the volume of a pyramid, since the base can be any multi-sided polynomial, then the variable B is used to denote the area of the base. For example, if the base of the pyramid was triangular, then the equation for the area of triangle would be used to determine B. Figure 2.46 below is an example of a pyramid where the base is a square, thus $B = w^2$.

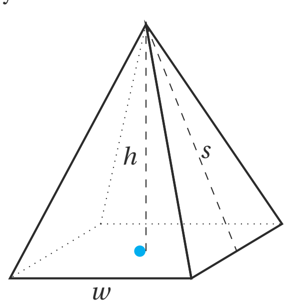


Figure 2.46: Right Pyramid

Volume and Surface Areas of Pyramids

The volume, V, lateral surface area, L, and total surface area, T of a pyramid is given by:

Figure:	Volume:	Lateral Area:	Total Area:
Pyramid	$V = \frac{1}{3}Bh$	$L = \frac{1}{2}ps$	$T = \frac{1}{2}ps + B$

where p is the perimeter of the base, s is the slant height, h is the altitude, and B is the area of the base.

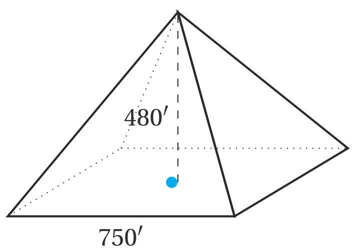


Figure 2.47: Great Pyramid of Giza

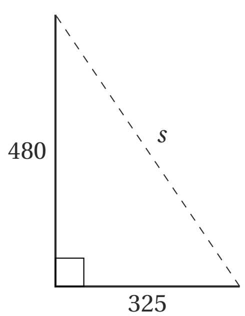


Figure 2.48

Example 2.4.3:

Find the volume and lateral surface area of the Great Pyramid of Egypt with an altitude 480 feet, and a base that is approximately square with a measurement of 750 feet on a side. Round results to the nearest hundredth

Solution:

To begin, everything is given to calculate the volume of the pyramid, thus the volume is:

$$\begin{aligned} V &= \frac{1}{3}Bh \\ &= \frac{1}{3}750^2(480) && \text{where } B = 750^2 \text{ is the area of the square base.} \\ &= 90\,000\,000.\ ft^3 \end{aligned}$$

The volume of the Great Pyramid of Giza is 90 million cubic feet. To determine the lateral surface area, we first have to find the slant height. Once again, we draw an illustration (figure 2.48) of what is known, and the variable we're looking for.

Using Pythagorean's theorem , $a^2 + b^2 = c^2$, we can solve for the slant height s.

$$\begin{aligned} s^2 &= a^2 + b^2 \\ s &= \sqrt{a^2 + b^2} \\ &= \sqrt{325^2 + 480^2} && \text{Do not perform intermediate rounding.} \end{aligned}$$

Now that we have s, the lateral surface area of the pyramid is

$$L = \frac{1}{2}(4)(750)\sqrt{325^2 + 480^2} \approx 869\,514.00\ ft^2$$

.

(Note: if we performed intermediate rounding of s to 2 decimal places, or even 4 places, the rounding error is significant enough to give us a different final result. Avoid intermediate rounding when possible.)

2.4.7 Frustrum

A frustrum can be in the shape of a *cone* or a *pyramid* where the shape appears to have been removed. The bases, top and bottom, of they frustrum are parallel as can be seen in figures 2.49 and 2.50. Since there are a limitless number frustrums in the shape of pyramid, we will focus on the frustrums in the shape of a cone.

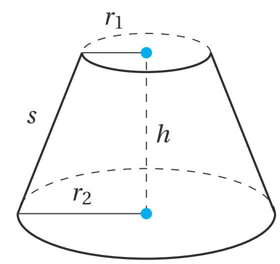


Figure 2.49: Cone Frustrum

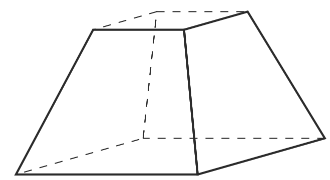


Figure 2.50: Pyramid Frustrum

Volume and Surface Areas of Circular Frustrum

The volume, V, lateral surface area, L, and total surface area, T of a pyramid is given by:

Figure:	Volume:	Lateral Area:	Total Area:
Frustrum	$V=\frac{h}{3}(B_1+B_2+\sqrt{B_1B_2})$	$L=\frac{s}{2}(p_1+p_2)$	$T=L+\pi(r_1^2+r_2^2)$

where p_1 and p_2 is the perimeter of the bases, s is the slant height, h is the altitude, and B_1 and B_2 are the area of the respective bases.

Example 2.4.4:

Find the volume, and the lateral surface area of a circular frustrum with radii measured at $r_1 = 1\ m$, $r_2 = 2\ m$, and has an altitude measured to be $h = 3\ m$.

Solution:

The volume should be straight forward since the given information is all that's required to evaluate the equation for volume. The Lateral surface area however, requires that we find the slant height as well.

$$
\begin{aligned}
V &= \frac{h}{3}(B_1+B_2+\sqrt{B_1B_2}) \\
&= \frac{3}{3}\left(\pi(1)^2+\pi(2)^2\right)+\sqrt{\pi^2(1)^2(2)^2} && \text{under the radical } \pi\cdot\pi \text{ is simplified to } \pi^2 \\
&\approx 21.99115 \\
&\approx 22\ m^3 && \text{round to the least precise number since the given components were measured and aren't exact.}
\end{aligned}
$$

We can use the pythagorean theorem to determine the slant height as long as the right triangle is drawn correctly (see figure 2.51).
Thus $s=\sqrt{3^2+1^2}=\sqrt{10}$. Now, the lateral surface area is:

$$
\begin{aligned}
L &= \frac{s}{2}(p_1+p_2) \;=\; \frac{s}{2}(2\pi r_1+2\pi r_2) \\
&= \frac{\sqrt{10}}{2}(2\pi(1)^2+2\pi(2)^2) && \text{avoid rounding } \sqrt{10} \text{ for now.} \\
&\approx 29.80376 \\
&\approx 30\ m^2 && \text{again round to the least precise number.}
\end{aligned}
$$

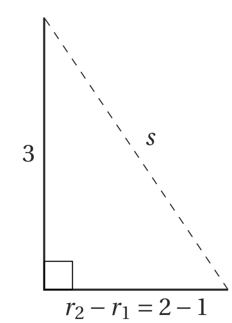


Figure 2.51

2.5 Similar Geometric Figures

In this section we concentrate on similar figures such as similar triangles, and similar volumes. By taking advantage of corresponding values to similar shapes, we can often simplify calculations that would otherwise be tedious. Also, in this section, we introduce a new concept called *proportion*. While we could spend a great deal of time on covering all facets of the concept of proportion, we are going to restrict our focus to adequately apply the concept to *similar geometric figures*.

A **proportion** is a relationship between one variable to another (length, quantity) who's ratio is a constant. For two ratios to be **proportional** to each other, their ratios must be equal. For example, we've already worked with proportions without explicitly stating it such as when we reduce a fraction: (i.e. $\frac{4}{8} = \frac{1}{2}$) We would say that 4 out-of 8 is equal 1 out-of 2. Since the ratios are the same we say the two quantities are proportional.

2.5.1 Continued Proportion

Note:
The proportion $x : y = a : b$ can be written as fractions two ways:

$$\frac{x}{y} = \frac{a}{b} \qquad \text{or} \qquad \frac{x}{a} = \frac{y}{b}$$

Notice that since x is proportional to a they both are in the numerator of the fraction. Same for y and b.

A proportion is commonly denoted as $x : y$ which is equivalent to $\frac{x}{y}$, or verbally we would say x out-of y. If we have two ratios that's denoted as $x : y = a : b$, then we call these two ratios *proportional*. A **continued proportion** involves six, or more, variables such that

$$x : y : z = a : b : c$$
$$\text{or equivalently}: \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

Basically this means that if $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, then $\frac{x}{a} = \frac{y}{b}$, $\frac{y}{b} = \frac{z}{c}$, and $\frac{x}{a} = \frac{z}{c}$. Continued proportions are used extensively throughout similar figures with regard to their lengths of sides, areas, volumes, angles, etc. It's important to note that there are extensions to the definition of *continued proportion* that this textbook does not include since they are not relevant to the focus here.

Example 2.5.1:

A 20 foot board is to be cut so that the lengths have a ratio of 13:8:5, what are the lengths of the sections of board? Round answer to the nearest 32^{nd} of an inch.

Solution:

Since each section of the board to be cut is a multiple of some unknown length, say x, then we have

$$\begin{aligned} 13x + 8x + 5x &= 20 \\ 26x &= 20 \\ x &= \frac{20}{26} \\ &\approx .769231\ ft \end{aligned}$$

The first board has a length of $13\left(\frac{20}{26}\right)$ ft., while the second has a length of $8\left(\frac{20}{26}\right)$ ft. and the third has a length of $5\left(\frac{20}{26}\right)$. However, we are asked to round the answers to the nearest 32^{nd} of an inch. There are several ways to approach this, but the most straight forward way is to begin with the decimal approximation in feet. Since $13\left(\frac{20}{26}\right) = 10.0$ feet, we'll use the second section of board to illustrate the unit conversion $8\left(\frac{20}{26}\right)$ is 6.153846154 ft. Now, we have 6 ft. and a fraction of a foot that needs to be converted to inches. So, take the fractional component and multiply it by 12, since there are 12 inches in a foot, to get 0.153846 ft$\frac{12\text{in}}{1\text{ ft}}$ = 1.846154 inches. Now we have 6′ 1.846154", and since we have little idea what 1.846154 inches is as a fraction we convert 0.846154 in to 32$^{\text{nd}}$ by use of proportions.

$$\frac{x}{32} = \frac{0.846154\text{ in}}{1}$$
$$x = 32(0.846154)$$
$$= 27.08\ 32^{\text{nd}}\text{ of an inch}$$

rounded to the nearest 32$^{\text{nd}}$ we have $\frac{27}{32}$. Finally, the second length is $6'\ 1\frac{27}{32}$". Ideally, all this calculation is done within the calculator in just a few keystrokes as shown in figure 2.52. The third measurement is shown in figure 2.53. Thus, we have the following lengths:

$$\text{length 1: } 10'\ 0''$$
$$\text{length 2: } 6'\ 1\frac{27}{32}''$$
$$\text{length 3: } 3'\ 10\frac{5}{32}''$$

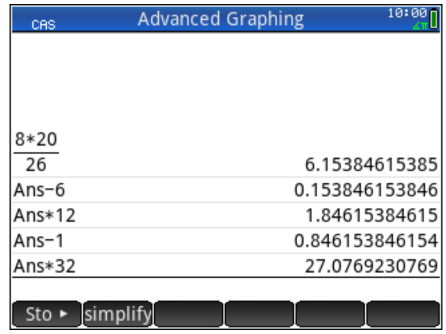


Figure 2.52

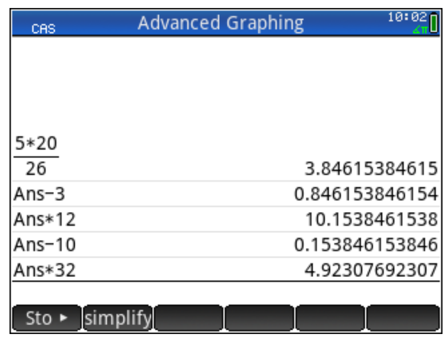


Figure 2.53

Example 2.5.2:

Find the values for the unknown variables: $x:3:5 = 4:y:15$

Solution:

It's often easier to write the ratios as fractions as opposed to the shorthand notation. Also setup each equation where only one unknown (variable) exists (if possible):

$$\frac{x}{4} = \frac{5}{15} \qquad \frac{3}{y} = \frac{5}{15}$$
$$x = \frac{4(5)}{15} \qquad 3 = \frac{5y}{15}$$
$$x = \frac{20}{15} \qquad 3(15) = 5y$$
$$x = \frac{4}{3} \qquad 5y = 45$$
$$y = \frac{45}{5}$$
$$y = 9$$

2.5.2 Similar Triangles

Similar triangles are two, or more, triangles that have congruent corresponding angles. The significance of similar triangles, or many similar shapes for that matter, is that the corresponding sides are proportional. This fact provides a very useful method for determining unknown sides that might otherwise be more difficult to solve. For example, figure 2.54 illustrates two *similar triangles* that has the following continued proportion:

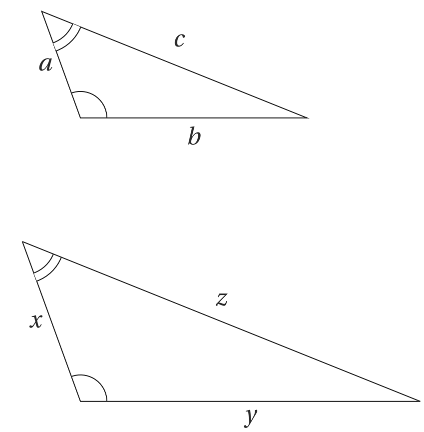


Figure 2.54: Similar Triangles

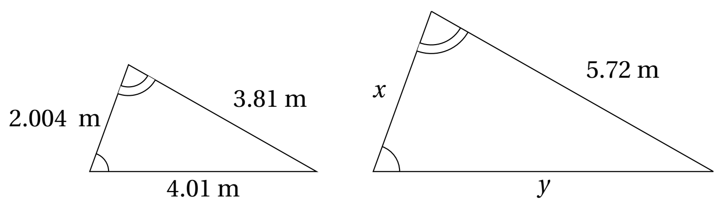
$$a : b : c = x : y : z \quad \text{or equivalently} \quad x : y : z = a : b : c$$

after rewriting in fractional form we have

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z} \quad \text{or} \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

Example 2.5.3:

Find the the values for x and y for the similiar triangles below. Measurements are not exact.



Solution:

Since the two similar triangles are proportional, then we must first match up the proportional sides: $x : 2.004$, $5.72 : 3.81$, and $y : 4.01$. The proportions can be set up as $\frac{2.004}{x} = \frac{3.81}{5.72}$, or $\frac{x}{2.004} = \frac{5.72}{3.81}$. Though, we'll reach the same result either way decide to set up the equation, it's simplest to set up ratios so that the unknown variable is in the numerator.

$$\frac{x}{2.004} = \frac{5.72}{3.81}$$

$$x = \frac{2.004(5.72)}{3.81}$$

$$\approx 3.01\ m$$

results are rounded to the least precise number.

When determining the other unknown variables, it's best to avoid results we just found such as $x = 3.01$ (if possible) for a couple reasons. One, there is always the possibility the our result is incorrect which will make all results dependent on it incorrect as well. Two, though not an issue with this problem, if the values given were exact, then it's best to use the exact values since our results may be approximated (rounded).

$$\frac{y}{4.01} = \frac{5.72}{3.81}$$

$$y = \frac{4.01(5.72)}{3.81}$$

$$\approx 6.02\ m$$

It can also be shown that the ratio of adjacent sides are congruent to the corresponding similar triangle. For example:

$$\frac{a}{x} = \frac{b}{y}$$

$$ay = bx \qquad \text{cross multiply}$$

$$\frac{a}{b} = \frac{x}{y} \qquad \text{divide both sides by } b \text{ and } y$$

Repeating this on all possible equations, then we have the following equivalent rational expressions:

$$\frac{a}{b} = \frac{x}{y} \qquad \frac{a}{c} = \frac{x}{z} \qquad \frac{b}{c} = \frac{y}{z}$$

The reciprocal is also true.

$$\frac{b}{a} = \frac{y}{x} \qquad \frac{c}{a} = \frac{z}{x} \qquad \frac{c}{b} = \frac{z}{y}$$

Example 2.5.4:

The illustration below represents a tower and a person standing on the shadow the tower makes. The person stands at a point so that the shadow cast by the person ends at the same location that the shadow ends cast by the tower. The person stands at 6 ft 2in tall, and his shadows length was measured at 10 ft 1in long. The length of the shadow cast by the tower was 389 ft. 2 in. long. How tall is the tower rounded to the nearest inch?

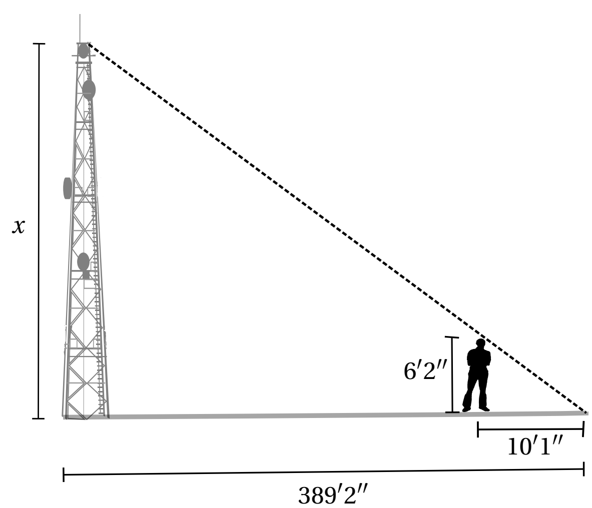


Figure 2.55

Solution:

The triangle that the tower makes and the person makes with respect to the lengths of their respective shadows creates a set of similar triangles. With the information given, there are several ways to set up our ratios to solve for the height of the tower. We'll do one way here, but see if you can find another ratio to get the same result.

$$\frac{\text{height of tower } (x)}{\text{height of man}} = \frac{\text{length of tower shadow}}{\text{length of man's shadow}}$$

$$\frac{x}{74} = \frac{4670}{121} \qquad \text{convert ft. to in.}$$

$$x = \frac{74(4670)}{121}$$

$$\approx 2856\ in \qquad \text{rounded to the nearest inch.}$$

The tower stands approximately 238 feet tall.

2.5.3 Similar Figures

Similar figures When any other figure, plane and solid figures, are similar, then the distance between any two points on one figure is proportional to the similar figure. The, *any* two points can represent any lengths such as radii, circumference, slant height, etc..

Example 2.5.5: – Similar Figures

Below are two similar cone frustrums, what is the slant height, s_2, of the larger frustrum, and the radius, R_1 of the larger frustrum?

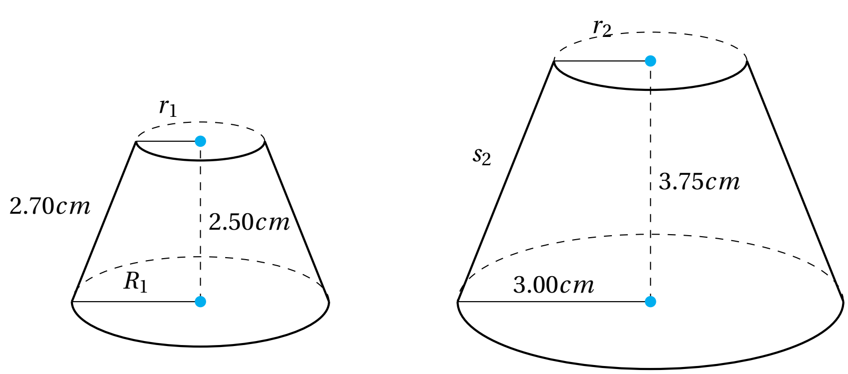


Figure 2.56

Solution:

Since these two figures are similar, then we know that all of their corresponding segments are proportional.

$$\frac{R_1}{R_2} = \frac{r_1}{r_2} = \frac{s_1}{s_2} = \frac{h_1}{h_2}$$

$$\frac{s_2}{h_2} = \frac{s_1}{h_1}$$
$$\frac{s_2}{3.75} = \frac{2.70}{2.50}$$
$$s_2 = \frac{3.75(2.70)}{2.50}$$
$$= 4.05\ cm$$

recall the zeros at the end of measurements are significant because they define the precision; thus we must round to the hundredths place.

To find R_1 we need to setup a similar ratio of corresponding segments.

$$\frac{R_1}{R_2} = \frac{h_1}{h_2}$$
$$\frac{R_1}{3.00} = \frac{2.50}{3.75}$$
$$R_1 = \frac{3.00(2.50)}{3.75}$$
$$= 2.00\ cm$$

note:
additional equivalent proportions can be set up by manipulating known proportions. For instance:

$$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$
$$r_1 h_2 = r_2 h_1$$
$$\frac{r_1}{h_1} = \frac{r_2}{h_2}$$

2.5.4 Areas and Volumes of Similar Figures

The ratio of the areas of any similar figures is equal to the ratio of the squares of any corresponding dimensions. For example, say we have two circles where one is larger than the other, thus we have the radius of the smaller as r_1, and r_2 for the larger, then the ratio of their areas is the following

$$\frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2}$$

While it's obvious to see how this simplifies to the ratio of the radii squared, the definition states that we can use any *corresponding dimensions*. Therefore, what would happen if we set the ratios of the areas equal to the ratios of the circumferences squared.

$$\frac{A_1}{A_2} = \frac{C_1^2}{C_2^2} = \frac{(2\pi)^2 r_1^2}{(2\pi)^2 r_2^2} = \frac{r_1^2}{r_2^2}$$

We, got the same result. In fact, we are not restricted to full dimensions of these figures either, rather we will still get the same result if we use a fraction of the circumference, or diameter, etc.

Example 2.5.6:

A floor plan for a house has a scale where 1/4 in. = 1 ft. A room on the floor plan has an area of 15 square inches. How many square feet does the actual room have?

Solution: Let A_2 represent the area for the actual room, while $A_1 = 15$ in^2 is the area for the room on the floor plan. 1/4 in. represents a fraction of the length of either the width or length of the room on the floor plan, and 1 ft. is a fraction of the length of the actual room's length or width. It doesn't matter whether we're referring to the length or width of the room, only that it is a corresponding dimension. Thus, we can set our ratio's up as the following:

$$\frac{\text{area of room } (A_2)}{\text{area of plan } (A_1)} = \frac{\left(\text{length of room}\right)^2}{\left(\text{length of plan}\right)^2}$$

$$\begin{aligned}
\frac{A_2}{A_1} &= \frac{1^2}{(1/4)^2} \\
\frac{A_2}{15} &= \frac{1^2}{(1/4)^2} \\
A_2 &= \frac{15(1)^2}{(1/4)^2} \\
&= 15(1)(16) \\
&= 240 \text{ ft}^2
\end{aligned}$$

The area of the actual room is 240 square feet.

Example 2.5.7:

Figure 2.57 illustrates two similar triangles where the smaller triangle has an area of 7 m^2, what is the area of the larger triangle?

Solution:

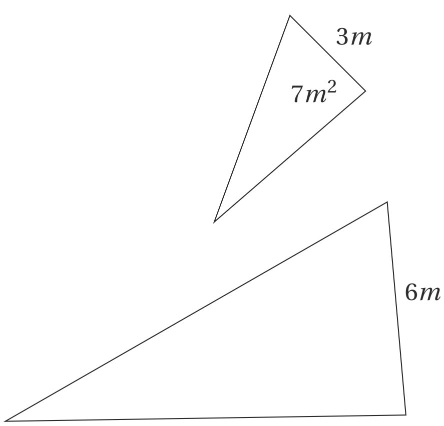


Figure 2.57

Since we are told the two triangles are similar, then we can determine the area of the larger triangle in the same way we solved the previous example. Letting A_2 and A_1 represent the areas for the larger and smaller triangles respectively, we have

$$\begin{aligned}
\frac{A_2}{A_1} &= \frac{6^2}{3^2} \\
\frac{A_2}{7} &= \frac{6^2}{3^2} \\
A_2 &= \frac{7(6)^2}{3^2} \\
&= \frac{252}{9} \\
&= 28 \ m^2
\end{aligned}$$

There is a similar relationship with regard to the **volumes of similar figures** where the ratio of corresponding volumes is equal to the ratio of corresponding dimensions cubed. For instance, a sphere of radius r_1 has volume equal to $(4/3)\pi r^3$, if we take the ratio of this sphere and another larger sphere with radius r_2 we get

$$\frac{V_1}{V_2} = \frac{(4/3)\pi r_1^3}{(4/3)\pi r_2^3} = \frac{r_1^3}{r_2^3}$$

Again, we are not restricted to just the radius, rather we can show this is true for any corresponding dimension of the sphere, or fraction of it, in the same way it was shown for areas.

Example 2.5.8:

In Example 2.5.5 we had two similar frustrums (shown in figure 2.58). If the volume of the smaller frustrum is 18.33 cubic centimeters, what is the volume of the larger frustrum?

Solution:

In this example, and based upon the information given, the only corresponding dimensions we have for both frustrums is the height.

$$\frac{V_2}{V_1} = \frac{h_2}{h_1}$$

$$\frac{V_2}{V_1} = \frac{3.75^3}{2.50^3}$$

$$\frac{V_2}{18.33} = \frac{3.75^3}{2.5^3}$$

$$V_2 = \frac{18.33(3.75)^3}{2.5^3}$$

$$= 61.86\ cm^3$$

The larger frustrum has a volume that is approximately 61.86 cubic centimeters.

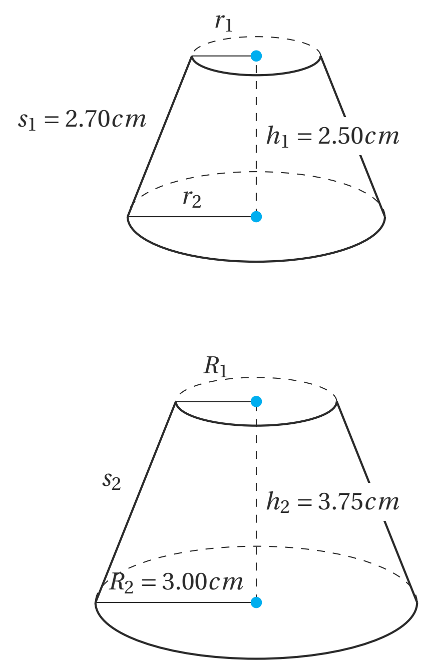


Figure 2.58

Chapter 3

Introduction to Trigonometry

3.1 Angles

In section 2.1 we introduced angles, and the different types of angles. In chapter 2 we stated that an angle is the amount of rotation of a ray about its vertex. In this chapter, it will be helpful to expand on the definition of an angle by assuming that the rays, or sides, used to define an angle aren't entirely arbitrary. For instance, when creating an angle we begin with an *initial side*, and end the angle at the *terminal side* as shown in figure 3.1. The main purpose of this modification to the definition of an angle is that we will often refer to the *sides* of an angle, and need to be able to distinguish between them.

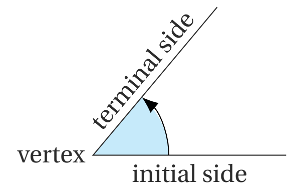


Figure 3.1

In this section, we will briefly review a few concepts of angles, and degree measure; however, for more details on these topics you should refer to section 2.1.

3.1.1 Positive and Negative Angles

Angles can be generated by rotating a ray about a fixed point. The starting position of the ray is called the **initial side** of the angle, while the final position of the ray is called the **terminal side** of the angle. The fixed point about which the ray is rotated is called the **vertex** of the angle. It is possible that the ray is rotated about the vertex by more than one full revolution. In reality, the ray can be rotated about the vertex as many times as we like as illustrated in figure 3.2. An angle generated by rotating a ray in the *clockwise* direction is defined as a **negative angle**, and if the rotation is *counterclockwise* the angle is a **positive angle** as can be seen in figure 3.3.

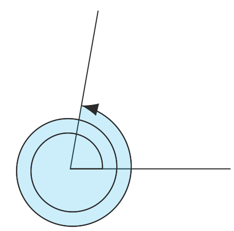


Figure 3.2: Angle greater than 360°

3.1.2 Degrees and Radians

There are two standard units of measurement that we use for describing the size of an angle: **degree** and **radian**. In degree measure, a complete circle is divided into 360°, thus one degree (denoted 1°) is an angle that is generated by rotating a ray $\frac{1}{360}$ of one complete revolution.

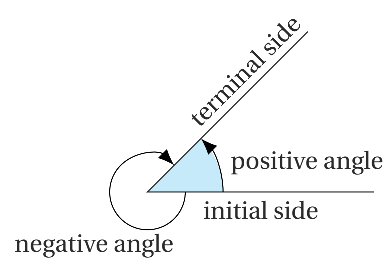


Figure 3.3

There are many practical purposes for which a degree is small enough to provide sufficient precision. However, when this is not the case and higher precision is needed, then the decimal system is often used. There are some applications where describing a fraction of an angle using decimals is not preferred. In this case, degrees are divided into sixty equal parts called **minutes**, and minutes are divided into sixty equal parts called **seconds**. This means that one minute (denoted $1'$) is 1/360 of a degree, and one second (denoted $1''$) is 1/360 of a minute or 1/3600 of a degree.

There are several hypothesis as to the origin of the base 60 numeral system, but its advantages are likely the motivation behind its use. For instance, first imagine a time before calculators and realize that 60 is the lowest number that is divisible by 1, 2, 3, 4, 5, and 6 making some calculations less tedious. When referring to angle measure, the base 60 numeral system called *sexagesimal* is often preferred in measuring time, angles, and geographical locations (latitude and longitude).

Example 3.1.1:

Express 16.1803° to an equal angle to the nearest second.

Solution:

To begin, recognize that the *minutes*, and *seconds* of an angle represent the fractional component, thus we know that the answer should be of the form $16^\circ\ x'\ y''$. We have to determine how many minutes are in 0.1803° first. To do this we multiply 0.1803 by 60.

$$x = 0.1803(60) = 10.818'$$

Since seconds represent a fraction of a minute, we calculate how many seconds are in $0.818'$ by multiplying by 60 again.

$$y = 0.818(60) = 49.08''$$

Thus, $16.1803^\circ = 16^\circ\ 10'\ 49''$

Some calculators have the ability to do these conversions for you; however, most do not. For this reason, and to better understand the nature of degrees, minutes, and seconds, it's important to know how to do these conversion by hand.

Sometimes it's preferable, or even necessary, to work with angles in decimal form only. Depending on how much rounding occurred, if any, during the conversion to degrees-minutes-seconds, this always adversely affects the precision of the angle. In the last example we rounded to the nearest minute while the original angle was precise to ten-thousandths of a degree; in the next example we see how much this affected our precision by converting back to decimal form.

Example 3.1.2:

Express $16.1803^\circ = 16^\circ\ 10'\ 49''$ to an equivalent angle rounded to the ten-thousandths place.

Solution:

To determine how many degrees 10 minutes is, we must divide by 60 which gives $10/60 = 0.1\overline{6}$. Since we are attempting to avoid intermediate rounding we'll need to use $10/60$ in fractional form because of its non-terminating decimal. Next, we need to determine how many minutes make up 49 seconds, before converting to degrees. We accomplish this by dividing 49 by 60 to get the number of minutes, then divide by 60 again to get the number of degrees. Alternative, yet equivalently, we could divide 49 by 3600. After putting adding both conversions together we get:

$$\begin{aligned} 16.1803^\circ = 16^\circ\ 10'\ 49'' &= 16^\circ + \left(\frac{10}{60}\right)^\circ + \left(\frac{49}{3600}\right)^\circ \\ &\approx 16.1803^\circ \end{aligned}$$

In this example, the rounding that occurred in the previous example was not significant enough to alter the conversion back to decimal form; however, this is not always the case.

Recall from section 2.1.3 that **radians** is a unit of measurement for an angle that is equal to the length of arc that it subtends. It is very important to understand the differences between degree measure and radian measure. Degree measure is simply a measure of angle without any regard to the length of the arc from which the angle is formed; however, radian measure is still a measure of an angle, but the units coincide with the length of arc of a circle of radius 1. So when should we use degrees, or radians? Typically we use the unit that the information is in; however, since degree measure has no reference to the length of arc, then anytime we are asked for length, velocity, accelration, etc. of an object moving in a circular motion then radians must be used.

To recall the equation needed to convert from degrees and radians, and vice versa, we only need to recall the proportion $180° = \pi rad$. From this proportion, we can create an equation to convert between the two units angle measurement.

Convert Between Radian and Degree Measure

Let d represent an angle in degrees, and r an angle in radians.

$$\frac{r}{\pi} = \frac{d}{180°}$$

Example 3.1.3:

Convert $60°$ to radians.

Solution:

Using the proportion defined above, we just need to substitute $60°$ for d and solve for r.

$$\frac{r}{\pi} = \frac{d}{180°}$$

$$\frac{r}{\pi} = \frac{60°}{180°}$$

$$r = \frac{\pi(60)}{180}$$

The degree symbol is a unit of measurement which cancels out.

$$r = \frac{\pi}{3}$$

$\frac{60}{180}$ reduces to $\frac{1}{3}$.

To convert from radians to degrees, we can use the same proportion and substitute our angle in radians for r, then solve for d. Recall that an angle measured in radians is often not identified as a denominate number; rather the absence of a degree symbol identifies the angle as radian measure. In addition, π seems to be almost synonymous with angles measures in radians; however, this is not always the case. It's often we deal with angles in decimal form only. Basically, if an angle is not identified with a degree symbol, then it is assumed to be in radian measure.

3.1.3 Coterminal Angles

A **coterminal angle** occurs when two angles share the same initial side and terminal side. Regardless of how many revolutions one angle makes, if the the terminal side is the same as the other angle, then they are coterminal. For example, if one angle is measured at 30°, and another angle is measured at 390°, then the two angles are coterminal. Figure 3.4 illustrates an angle measured at 30°, and another angle measuring 390°. Figure 3.5 shows two coterminal angles measuring 120° and −240°.

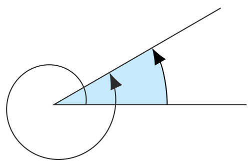


Figure 3.4: Coterminal Angels

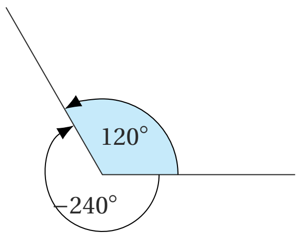


Figure 3.5

Given any angle, to find another angle that is coterminal we only need to either add, or subtract 360° ($\pm 2\pi$ if in radians), or any multiple of it. For instance, we know that 30° and 390° are coterminal, then $30° + 5(360°) = 1830°$ is also coterminal.

For homework purposes, if you are given an angle and asked to find both a positive, and negative coterminal angles for it, then only add 360°, or 2π if in radians, for the positive angle. Subtract 360°, or -2π if in radians, for the negative coterminal angle. The reason for this is that there are infinitely many coterminal angles for any given angle, and we must narrow the possibilities for the sake of right and wrong answers.

3.1.4 Angles in Standard Position

Technically an angle can be defined with its initial side and terminal side oriented in any position you like. However, an angle defined in **standard position** is oriented so that the initial side coincides with the positive x-axis, and the vertex is located at the origin. Figure 3.6 shows an angle of 60° drawn in standard position.

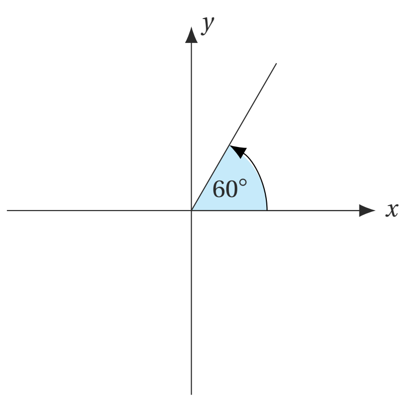


Figure 3.6: Standard Angle at 60°

When working within the **Euclidean plane**(two-dimensional space), or the more common term **Cartesian** coordinate plane,we often refer to **quadrants** as quadrant I, II, III, and IV. Beginning with quadrant I located where both x, and y values are positive, we label the other quadrants in a counter-clockwise rotation. Below, in figure 3.7, each of the four quadrants are shown.

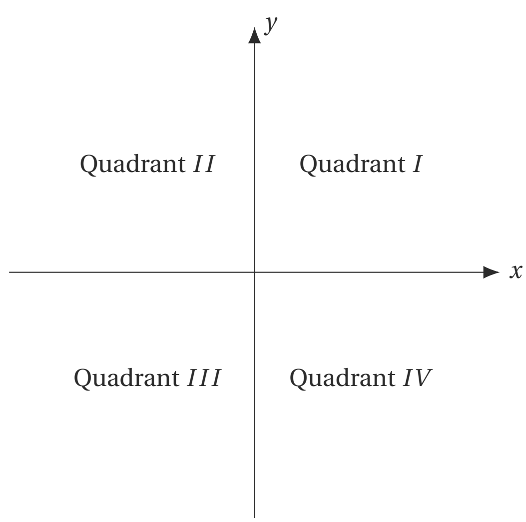


Figure 3.7

3.2 Trigonometric Functions

3.2.1 Trigonometric Functions

For an angle in standard position, there are six possible ratios for the sides of the triangle formed with respect to the angle θ. Figure 3.8 illustrates a right triangle formed from an angle in standard position with radius r, and angle θ. Notice that the radius is arbitrary, this is because regardless of the length of r, or any point (x, y), along the ray in the direction defined by θ, these six ratios will always have the same value. On the other hand, the following six ratios are entirely dependent on the angle θ, and will give different values as θ is changed. If θ and another angle, say ϕ, are coterminal angles, then the ratios will be the same.

The following six trigonometric function are defined based on the illustration of figure 3.8.

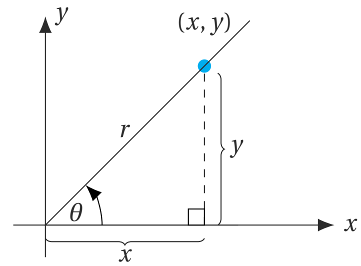


Figure 3.8

Trigonometric Functions

Sine of θ: $\sin\theta = \dfrac{y}{r}$ Cosecant of θ: $\csc\theta = \dfrac{r}{y}$

Cosine of θ: $\cos\theta = \dfrac{x}{r}$ Secant of θ: $\sec\theta = \dfrac{r}{x}$

Tangent of θ: $\tan\theta = \dfrac{y}{x}$ Cosine of θ: $\cot\theta = \dfrac{x}{y}$

Example 3.2.1:

Find the six trigonometric functions if the point $(9, 12)$ lies on the terminal side of an angle drawn in standard position. Leave answers in exact form.

Solution:

Drawing an illustration and labeling the information given usually helps, but is not necessary. Figure 3.9 shows the information given with labels. In order to state the six trigonometric functions, we will have to find the length of the hypotenuse first. This can be done using the Pythagorean theorem which is covered in section 2.1.

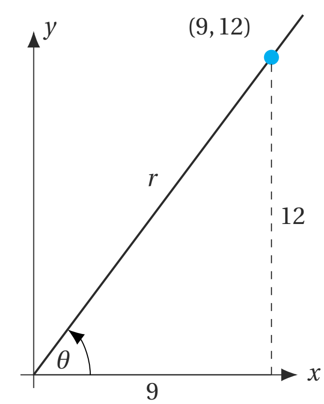


Figure 3.9

$$r = \sqrt{9^2 + 12^2} = \sqrt{225} = 15$$

Substituting with $x = 9$, $y = 12$, and $r = 15$ we have

$$\sin\theta = \frac{12}{15} = \frac{4}{5} \qquad \csc\theta = \frac{15}{12} = \frac{5}{4}$$

$$\cos\theta = \frac{9}{15} = \frac{3}{5} \qquad \sec\theta = \frac{15}{9} = \frac{5}{3}$$

$$\tan\theta = \frac{12}{9} = \frac{4}{3} \qquad \cot\theta = \frac{9}{12} = \frac{3}{4}$$

Since a point can be defined anywhere within the plane, we will often encounter points where either the x-coordinate, y-coordinate, or both are negative. Nevertheless, the hypotenuse of the triangle formed by any point (x, y) is never negative. Looking back at the Pythagorean theorem we see that $r = \sqrt{x^2 + y^2}$ can never result in a negative value.

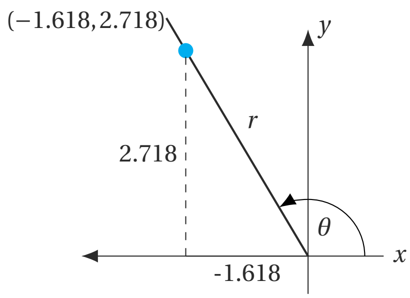


Figure 3.10

Example 3.2.2:

Find the six trigonometric functions if the point $(-1.618, 2.718)$ lies on the terminal side of an angle drawn in standard position. Leave answers to the nearest thousandth.

Solution:

Just as we did in the last example, we have to find r using the Pythagorean theorem first.

$$r = \sqrt{(-1.618)^2 + 2.718^2} \approx 3.16314$$

Since r has resulted in a possible non-terminating decimal, we'll have to round the result. In an attempt to avoid too much error due to intermediate rounding, we've rounded r to two more decimal places than the final result requires.

Now that we have r, the six trigonometric functions are:

$$\sin\theta = \frac{2.718}{3.16314} = 0.859 \qquad \csc\theta = \frac{3.16314}{2.718} = 1.164$$

$$\cos\theta = \frac{-1.618}{3.16314} = -0.512 \qquad \sec\theta = \frac{3.16314}{-1.618} = -1.955$$

$$\tan\theta = \frac{2.718}{-1.618} = -1.680 \qquad \cot\theta = \frac{-1.618}{2.718} = -0.595$$

The examples shown in this section refer to the length of r as simply the radius, or hypotenuse of the triangle. The length of r is typically referred to as the **radius vector**. Later, in chapter **??**, we discuss vectors in greater detail.

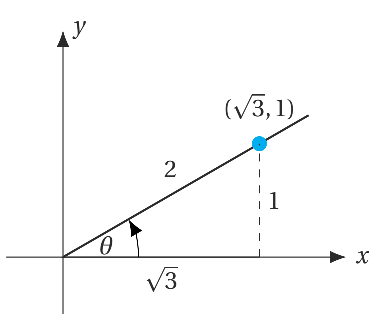


Figure 3.11

Example 3.2.3:

Given that the $\sin\theta = 0.5$, find the remaining trigonometric functions.

Solution:

Again, the best way to begin is to draw a diagram of the information given, and label what is known. Since $\sin\theta = \frac{x}{r}$, then we have two approaches that we could take for this problem. The first is to convert the decimal 0.5 into a fraction of $\frac{1}{2}$ where $x = 1$, and $r = 2$. The other approach we could take is to let $x = 0.5$, and $r = 1$ so that the fraction looks like $\frac{0.5}{1}$. Either method will give the same result. Here, we'll go with the first approach; however, you should try the second on your own to verify the results are the same.

Using the Pythagorean theorem we find that $x = \sqrt{2^2 - 1^2} = \sqrt{3}$, and can now list the remaining trigonometric functions:

$$\sin\theta = \frac{1}{2} \qquad \csc\theta = \frac{2}{1} = 2$$

$$\cos\theta = \frac{\sqrt{3}}{2} \qquad \sec\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

3.2.2 Rationalizing the Denominator

You'll may have noticed that the radicals in the denominator, in the last example, were simplified so that the radical is in the numerator. This process of simplification is called **rationalizing** the denominator. There are several reasons for doing this which go back to when arithmetic was done by hand; however, today it's just good practice to maintain a standard. Take a look at the example below, and ask yourself which you would rather calculate by hand.

$$\frac{1}{\sqrt{3}} \approx \frac{1}{1.7320500808...} \qquad \frac{\sqrt{3}}{3} \approx \frac{1.7320500808...}{3}$$

Even using todays technology computers would begin to lose precision with an irrational number in the denominator, albeit probably not by the precision in which we're interested in. Regardless, all answers in both the textbook, and on-line will use the rationalized form when applicable.

To rationalize the denominator, we must first understand that we can only multiply a number, or fraction, by a value that's equivalent to 1. In the instance above for $\frac{1}{\sqrt{3}}$, the value that was multiplied was $\frac{\sqrt{3}}{\sqrt{3}}$. From there we use properties of roots and exponents from section 1.3 to simply the expression.

$$\begin{aligned}\frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}} \\ &= \frac{\sqrt{3}}{\left(\sqrt{3}\right)^2} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

you may recognize that squaring a square root cancels each other leaving only the 3, but if not then continue to use properties.

3.3 Values of Trigonometric Functions

In the last section we found out how to get the trigonometric values if given a point in the plane, or if given one other trigonometric function value. In this section we look at the function values for specific angles of θ, some trigonometric identities, and trigonometry using the calculator.

3.3.1 Exact Trigonometric Function Values

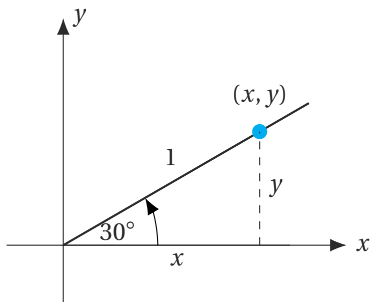


Figure 3.12

There are only a handful of trigonometric values in which we can easily determine their exact values. Specifically, the values correspond to angles that are multiples of 30°, and multiples of 45°. To begin, lets take a look at the trigonometric values for 30°. However, in an attempt to simplify the arithmetic, we'll assume the length of the *radius vector* is 1. If you recall from the previous section 3.2, the length of the radius vector is not a concern since the ratios for the trigonometric functions always result in the same value, thus letting $r = 1$ is just as arbitrary as letting $r = \sqrt{2}$.

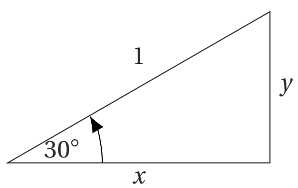


Figure 3.13

Before we begin, take a moment to study figure 3.12. You'll notice that aside from knowing the angle, and the length of the radius vector that we don't have much else to work with (at least for now) at first glance. To determine the exact values for both x, and y, it may be easier if we redraw the triangle without some clutter as shown in figure 3.13.

If we take advantage of geometry, we can deduce at least one side of the triangle. For instance, redraw another triangle that is a mirror image about the x-axis of the one shown in figure 3.13. When we do this, we get an equilateral triangle as shown in figure 3.14. An equilateral triangle has all sides of the same length. Since we know the length of r to be 1, then all sides of the equilateral triangle is 1. Notice that there are two y values in this illustration; this means that one of the y values must be equal to $1/2$. If we redraw our first triangle with $y = 1/2$ (Figure 3.15), then the use of the Pythagorean theorem will give us the value for x.

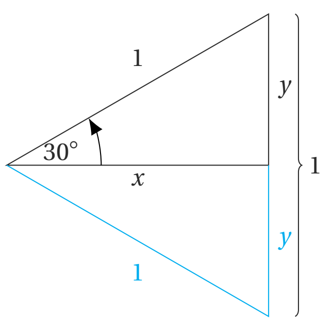


Figure 3.14

$$x = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$$

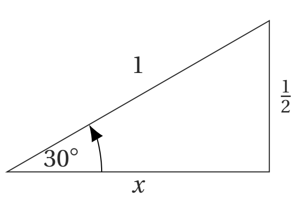


Figure 3.15

For a 60° triangle the arithmetic and logic are the same except the values for x and y will be swapped. I encourage you to verify this on your own, but for convenience the 60° triangle is shown below. Note that all colored labels were not known to begin with.

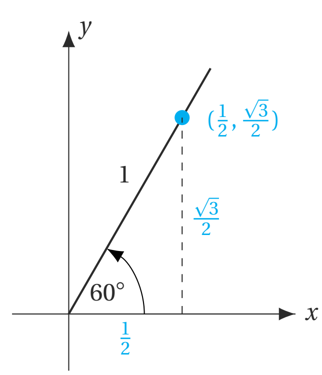


Figure 3.16

For a 45° angle with $r = 1$, the logic is more straight forward. Figure 3.17 shows an illustration of what is known. Similarly, things are a little easier to follow by redrawing the triangle showing only the information that's needed (figure 3.18). Since the angle opposite the hypotenuse is a right angle (90°), and the sum of all angles in any triangle is 180°, then the remaining angle must be 45° as well. We know that if the two angles have the same measure in a triangle, then its called an *isosceles* triangle. Consequently, the two adjacent sides must also be the same which means that for whatever value x is, the value for y is the same. Redrawing the illustration with this new information is shown in figure 3.19. From here the Pythagorean theorem can determine the value for x when we substitute $y = x$.

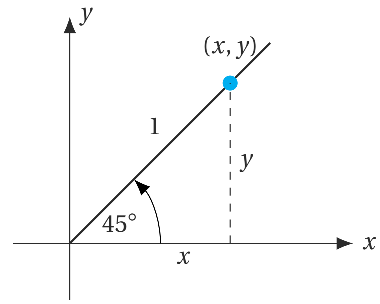


Figure 3.17

$$
\begin{aligned}
x^2 + y^2 &= r^2 \\
x^2 + x^2 &= 1^2 && \text{substitute for } r = 1\text{, and } y = x. \\
2x^2 &= 1 \\
x^2 &= \frac{1}{2} \\
x &= \pm\sqrt{\frac{1}{2}} \\
&= \frac{\sqrt{1}}{\sqrt{2}} && \text{we're only interested in the positive radical, so drop the } \pm \text{ as no sign implies positive.} \\
&= \frac{1}{\sqrt{2}} && \text{while this is correct, we want to rationalize the denominator by multiplying by } \tfrac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\end{aligned}
$$

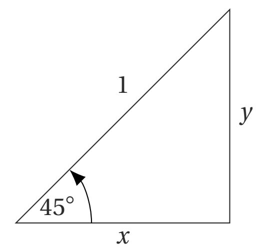


Figure 3.18

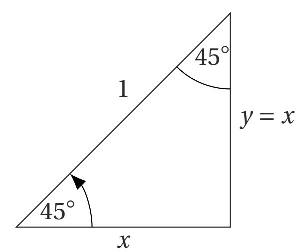


Figure 3.19

Thus, the exact values for a 45° angle are $x = \frac{\sqrt{2}}{2}$, and since $y = x$, then $y = \frac{\sqrt{2}}{2}$ as well. The illustration for the 45° angle in standard position is shown in figure 3.20.

So far, we've only seen the exact values for angles of 30°, 60°, and 45°; however, we stated that we can determine the exact values for any angle that was a multiple of these angles. Because of the **symmetric** nature of a circle with a fixed radius, meaning the lengths of x and y are the same when reflected about the axes, the values for multiples of these angles will be the same except for possible changes in signs. In addition, multiples of these angles that lie on an axis (multiples of 90°), either x-axis or y-axis, will have different values as well; specifically, the values of x and y will be either a zero or $\pm r$ depending on the angle θ.

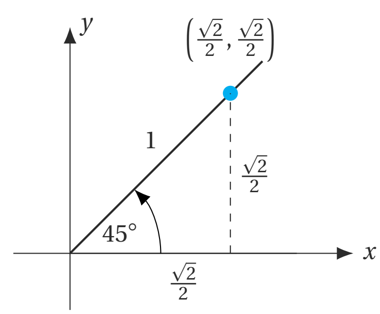


Figure 3.20

Since the trigonometric function values are ratios of the sides of a triangle, it's not necessary to solve for the exact values as the radius changes. For instance, if you're asked to find the exact value for the $\sin 30°$ of a triangle that has a hypotenuse of length 10, then the answer will be the same as that of triangle with a hypotenuse of length 1 which is 1/2 as we've solved earlier. You are encouraged to commit these few trigonometric values, shown on the unit circle presented in subsection 3.3.2, to memory as they are used often.

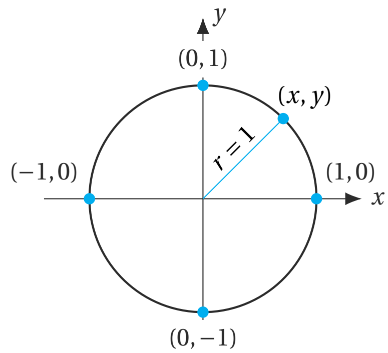


Figure 3.21

3.3.2 The Unit Circle

The **unit circle** is a circle of radius 1 (one) centered at the origin (0,0). Often in trigonometry, the unit circle is depicted with angle measures, and their respective *Cartesian coordinates* for known exact values (figure 3.22). **Cartesian coordinates** is a system that specifies each point uniquely in a plane by a pair of signed numerical values, which are fixed distances to the point from the axes (in short, the point (x, y) you're used to). Figure 3.21 shows the unit circle with an arbitrary point and arbitrary radius used for reference.

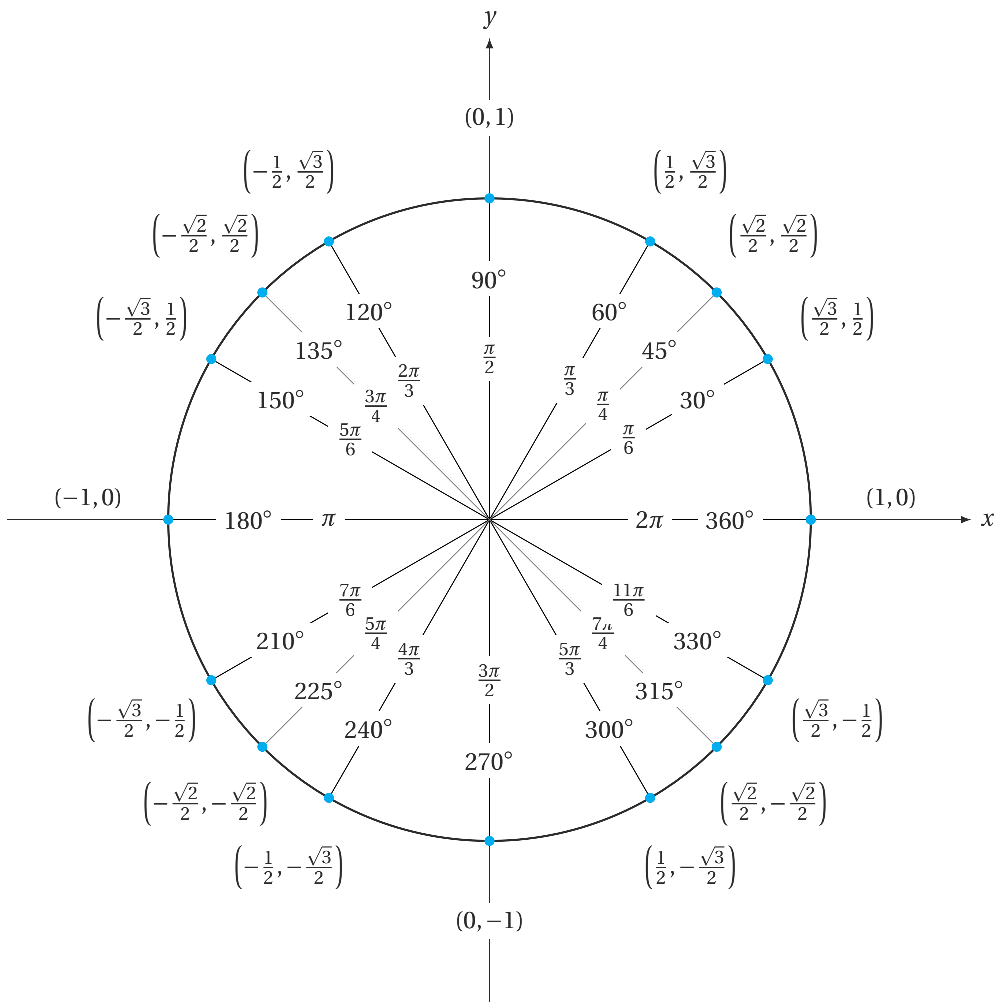


Figure 3.22: Unit Circle

Trigonometric Functions for the Unit Circle

Sine of θ:	$\sin\theta = y$	Cosecant of θ:	$\csc\theta = \dfrac{1}{y}$
Cosine of θ:	$\cos\theta = x$	Secant of θ:	$\sec\theta = \dfrac{1}{x}$
Tangent of θ:	$\tan\theta = \dfrac{y}{x}$	Cosine of θ:	$\cot\theta = \dfrac{x}{y}$

Looking at figure 3.22 notice the symmetry of the points around the circle. Considering that 30° and 60° angles have the same, yet reversed, values; and the 45° angle has the same values for both x and y makes recalling these values less intimidating. As stated before, many calculators do not display answers in exact format; therefore, these values would have to be committed to memory.

Example 3.3.1:

Using the unit circle, find the $\sin\theta$, $\cos\theta$, and $\tan\theta$ where $\theta = 45°$

Solution:

Since $x = \cos\theta$, $y = \sin\theta$, then both x and y are $\frac{\sqrt{2}}{2}$. Thus we have

$$\cos 45° = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin 45° = \frac{\sqrt{2}}{2}$$

Since $\tan\theta$ is defined as the ratio of y/x, then

$$\tan 45° = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

For the next couple examples you may find it useful to review *coterminal angles* which can be found in subsection 3.1.3 on page 89.

Example 3.3.2:

Find the six trigonometric functions for $\theta = \frac{-2\pi}{3}$.

Solution:

First, notice that $\theta = \frac{-2\pi}{3}$ is not shown explicitly on the unit circle; however, it is coterminal with $\theta' = \frac{-2\pi}{3} + 2\pi = \frac{4\pi}{3}$ which has the coordinate $(\frac{-1}{2}, \frac{-\sqrt{3}}{2})$. Using the definitions of the trigonometric functions in section 3.2, we have the following:

$$\begin{aligned}
y &= \sin\frac{4\pi}{3} = \frac{-\sqrt{3}}{2}, & \frac{1}{y} &= \csc\frac{4\pi}{3} = \frac{2}{-\sqrt{3}} = \frac{-2\sqrt{3}}{3} \\
x &= \cos\frac{4\pi}{3} = \frac{-1}{2}, & \frac{1}{x} &= \sec\frac{4\pi}{3} = \frac{2}{-1} = -2 \\
\frac{y}{x} &= \tan\frac{4\pi}{3} = \frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}} = \sqrt{3}, & \frac{x}{y} &= \cot\frac{4\pi}{3} = \frac{\frac{-1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}
\end{aligned}$$

Example 3.3.3:

Find the values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ of the angle $\theta = \frac{13\pi}{6}$.

Solution:

As in the last example, $\theta = \frac{13\pi}{6}$ is not explicitly stated in the unit circle. Nevertheless, it is a coterminal angle, but which one? To find the coterminal angle, rewrite $\frac{13\pi}{6}$ as a mixed number.

$$
\begin{aligned}
\frac{13\pi}{6} &= \left(\frac{13}{6}\right)\pi \\
&= \left(2\frac{1}{6}\right)\pi \\
&= \left(2+\frac{1}{6}\right)\pi \\
&= 2\pi + \frac{\pi}{6}
\end{aligned}
$$

Thus, the angle makes one complete revolution and another $\frac{\pi}{6}$ which is our coterminal angle.

$$
\begin{aligned}
\sin\frac{13\pi}{6} &= \sin\frac{\pi}{6} = \frac{1}{2} \\
\cos\frac{13\pi}{6} &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\
\tan\frac{13\pi}{6} &= \tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}
\end{aligned}
$$

Note:
There are several ways to determine coterminal angles. Another method to determining the coterminal angle in Example 3.3.3 is to subtract 2π from the angle given in this case.

$$
\begin{aligned}
\frac{13\pi}{6} - 2\pi &= \frac{13\pi}{6} - \frac{12\pi}{6} \\
&= \frac{\pi}{6}
\end{aligned}
$$

3.3.3 Reciprocal Trigonometric Identities

Trigonometric Functions

Sine of θ:	$\sin\theta = \frac{y}{r}$	Cosecant of θ:	$\csc\theta = \frac{r}{y}$
Cosine of θ:	$\cos\theta = \frac{x}{r}$	Secant of θ:	$\sec\theta = \frac{r}{x}$
Tangent of θ:	$\tan\theta = \frac{y}{x}$	Cosine of θ:	$\cot\theta = \frac{x}{y}$

If we refer back to the definitions of the six trigonometric functions which are shown above for convenience, you may begin to notice a pattern where some of the trigonometric functions appear to be the reciprocal of another. If you have, then you are right and these functions are called the **reciprocal identities**. If this is not apparent, then lets take a look at the sine function. Since $\sin\theta$ is defined as $\sin\theta = \frac{y}{r}$, then $y = r\sin\theta$. If we replace y in the cosecant ($\csc\theta$) with $y = r\sin\theta$ then we get:

$$\csc\theta = \frac{r}{y}$$
$$\csc\theta = \frac{r}{r\sin\theta} = \frac{1}{\sin\theta}$$

If we do this for the remaining trigonometric functions then we have the following reciprocal identities.

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Following this same logic, we can see a relationship between $\sin\theta$, $\cos\theta$, and $\tan\theta$ which are known as the **quotient identities**.

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

3.3.4 Trigonometric Functions on the Calculator

As we stated before, there are only a handful of angles in which the exact trigonometric values can be determined which we often refer to as simply **special angles**. The rest of the angles must be approximated by use of a calculator. Depending on the calculator used, the entry method can differ. For example, if using a *single line* display calculator the angle must be entered first before calling the trigonometric function wanted. As for the *multi-line* calculator (the type we'll show), the entry method is the same as it's displayed on paper. Some calculators have the ability to display answers in symbolic form such as our *special angles*; however, many do not. Figure 3.23 shows the display of $\cos 30°$ in both symbolic, and approximated/decimal form.

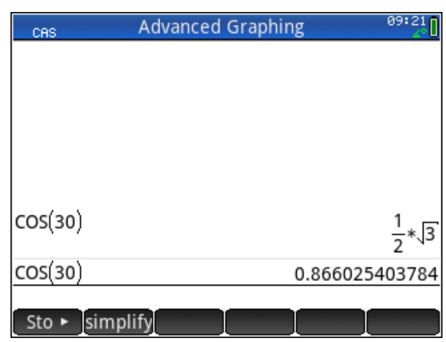


Figure 3.23

By this point, it's important that you have a good understanding of what each trigonometric function represents, and the differences between degree measure and radian measure. Most calculators are unique when it comes to accessing certain settings such as changing degree modes. You'll need to refer to your owners manual to become familiar with how to operate your particular calculator. It's also important to know that most calculators only have functions for sine, cosine, and tangent along with their respective inverse functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ which we cover shortly. Thus, functions for $\sec\theta$, $\csc\theta$, and $\cot\theta$ are not typically available. Nevertheless, these three functions are easily determined in the calculator since they are reciprocals of $\sin\theta$, $\cos\theta$, and $\tan\theta$ respectively.

Example 3.3.4: – Entering Trig Functions in the Calculator

Find the six trigonometric function values for 37.08° precise to the hundredths place.

Solution:

Since 37.08° is not one of our special angles, then we'll have to use the calculator to approximate the values. Beginning with sine, cosine, and tangent we have the following calculator output shown in figure 3.24.

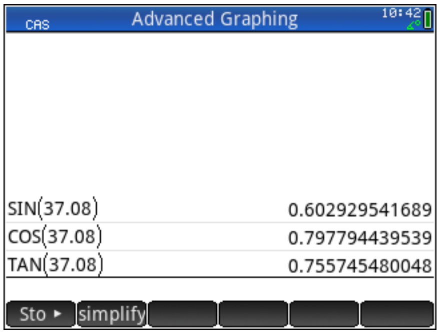


Figure 3.24

$$\sin\left(37.08°\right) = 0.60 \quad \cos\left(37.08°\right) = 0.80 \quad \tan\left(37.08°\right) = 0.76$$

To find the values for csc(37.08°), sec(37.08°), and cot(37.08°) we have to enter them as the reciprocal of sin(37.08°), cos(37.08°), and tan(37.08°) respectively as shown in figure 3.25.

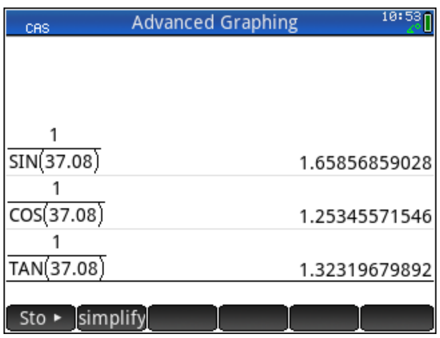


Figure 3.25

$$\csc\left(37.08°\right) = \frac{1}{\sin(37.08°)} = 1.66$$

$$\sec\left(37.08°\right) = \frac{1}{\cos(37.08°)} = 1.25$$

$$\cot\left(37.08°\right) = \frac{1}{\tan(37.08°)} = 1.32$$

Notice that the entry method in the HP calculator used in this text is displayed in the same *textbook* manner. This entry method is not uncommon with most multi-line scientific calculators.

Example 3.3.5: – Entering Trig Functions in the Calculator

Find the value for $\sin(1.618\pi)$ accurate to four significant digits.

Solution:

This calculation is the same as that in the previous example with the exception that the mode must be changed to radians. Each calculator is different; however, if you are using the HP Prime, or its emulator, then the fastest way to change modes is to touch the top right portion of the screen where it shows an angle symbol. When you touch that portion of the screen, a small window will pop up allowing you to select the symbol $\boxed{\measuredangle\pi}$ for radian mode. After changing modes, we enter the function just as it is printed and select $\boxed{\text{Enter}}$. The result is shown in figure 3.26.

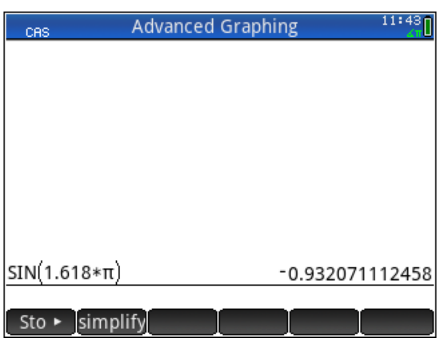


Figure 3.26

$$\sin(1.618\pi) = -0.9321$$

Note: Reciprocal identities such as $\csc\theta$ can be entered as $\frac{1}{\sin\theta}$, or $(\sin\theta)^{-1}$. Keep in mind that $\sin^{-1}\theta \neq (\sin\theta)^{-1}$.

Example 3.3.6: – Entering Reciprocal as x^{-1}

Find csc(2.7183) correct to 3 significant digits.

Solution:

First, note that the value 2.7183 is not identified as a *denominate number* (no unit of measurement identified), and in this case the value 2.7183 is automatically assumed to be in radians (not all radian measures have the symbol π associated with it).

Though the directions do not specify how to calculate the answer, we'll take a different approach when entering the function in the calculator. The following are the key strokes used:

(sin 2.7183) x^y (+/- 1) Enter

First, note that the exponent key can appear as the key symbol x^y, or ∧ depending on the calculator used. If entered correctly you should get the following result.

$$\csc(2.7183) = (\sin(2.7183))^{(-1)} = 2.434$$

3.3.5 Inverse Trigonometric Functions

Until now, we have determined the values of the trigonometric functions given an angle, or two or more sides. What if we were given two, or more, sides and want to know the angle? The inverse trigonometric function keys on the calculator is what is used to determine the angles of a right triangle. The inverse trigonometric function keys typically appear as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ on most calculators and are pronounced **inverse sine**, **inverse cosine**, and **inverse tangent** of a respectfully. Due to the notation confusion of, say $\sin^{-1} a$, inverse functions where the notation is easily confused with the reciprocal function, say $(\sin\theta)^{-1}$, some calculators use an abbreviated form of the **arcus** trigonometric functions such as asin, acos, and atan which refer to **arcsine**, **arccosine**, and **arctangent** of a respectfully. These arcus functions are simply another name for the inverse trigonometric functions.

Example 3.3.7:

What is the angle measure for θ in figure 3.27 rounded to hundredths of a degree?

Solution:

Since we already know the lengths of all the sides, then we could use any of the inverse trigonometric functions to determine θ. In this case, since one function is just as arbitrary as the next, we'll show all three inverse trigonometric functions to find θ; and verify that the results are the same.

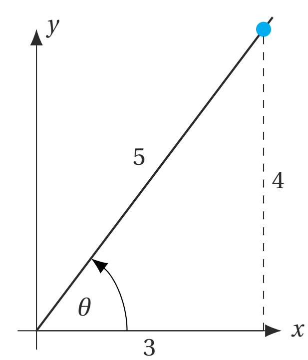


Figure 3.27

$$\sin\theta = \frac{y}{r} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{y}{r}\right)$$
$$\sin\theta = \frac{4}{5} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{4}{5}\right) = 53.13^\circ$$

$$\cos\theta = \frac{x}{r} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{x}{r}\right)$$
$$\cos\theta = \frac{3}{5} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{3}{5}\right) = 53.13^\circ$$

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\tan\theta = \frac{4}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

Thus, all three inverse functions gave the same result as we expected.

Example 3.3.8:

Find the angle for the trigonometric function $\tan\theta = 1.532$ rounded to two decimal places in degrees .

Solution:

It's not necessary to rewrite 1.532 as a fraction when using inverse functions.

$$\begin{aligned} \tan\theta &= 1.532 \\ \theta &= \tan^{-1} 1.532 \\ &= 56.87^\circ \end{aligned}$$

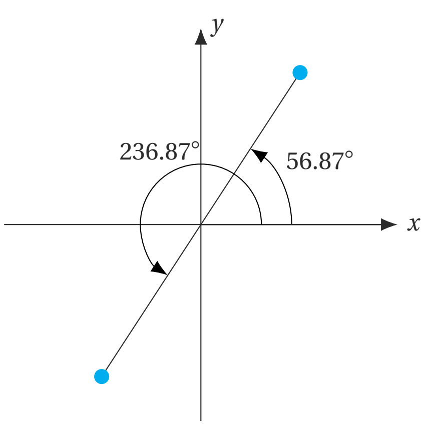


Figure 3.28

Inverse trigonometric functions are used for determining angles; however, you must also be familiar with which angle you want. In the above example, we were asked to determine θ for $\tan\theta = 1.532$ which we did, but what if we knew the angle resided in the 3^{rd} quadrant. In the third quadrant both x, and y values are negative and the ratio of two negative numbers would be positive, thus it's entirely possible that the coordinate was in $Q1$ or $Q3$. If we knew the angle is in $Q3$, then we would have to add 180° to the answer the calculator gave (These two angles are illustrated in figure 3.28). The reason for this is comes down to the *range* of values for which $\tan^{-1}$ is defined; however, a deeper understanding of functions and their inverses is needed to explain further which we will not go into in this chapter. For now, it's sufficient to recognize the quadrant you're in, and to know when to add 180°, and when not to.

The easiest method is to know which quadrant the angle is in, and the the range of values for which the calculator will give you answers. For instance, for $\theta = \sin^{-1}$, the calculator will only give results from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, or -90° ot 90°, for $\theta = \cos^{-1}$ the calculator give values between 0 and π, or 0° and 180°, and $\theta = \tan^{-1}$ the calculator gives values between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, or -90° to 90°. Below is a graphical representation (figure 3.29) for the range of values for each of the inverse trigonometric functions just described.

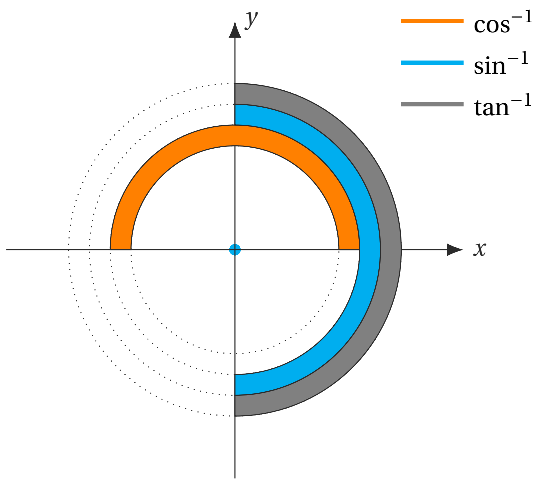


Figure 3.29: Ranges of Inverse Trigonometric Functions

Example 3.3.9:

Find the angle (in degrees) of the line that begins at the origin and has its terminal point at $(-3,-4)$.

Solution:

First, note that the terminal point lies in quadrant III since both x and y coordinates are negative. This means that we'll have to add 180° to the angle given from the calculator.

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ \theta &= \tan^{-1}\left(\frac{y}{x}\right)\\ &= \tan^{-1}\left(\frac{-4}{-3}\right)\\ &= 180^\circ + \tan^{-1}\left(\frac{-4}{-3}\right) && \text{add 180° since the calculator will intepret } \tan^{-1}\left(\tfrac{-4}{-3}\right) \text{ as } \tan^{-1}\left(\tfrac{4}{3}\right)\\ &\approx 233.13^\circ\end{aligned}$$

3.4 The Right Triangle

3.4.1 Parts of a Right Triangle

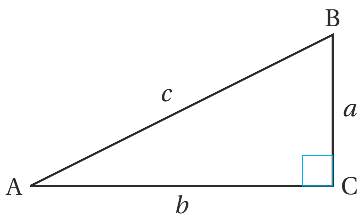


Figure 3.30

To solve for the remaining components (sides and angles) of any triangle we must be given at least three parts where one of those parts must include at least one side. Having all angles of a triangle tells us nothing about the length of the sides of a triangle. The side opposite the 90° angle is always the longest side of the right triangle and is called the **hypotenuse**. The other two sides are referred to as the **legs**. When using Latin letters to denote the parts of a triangle we use uppercase letters to reference the angles such as $(A, B, C, ...)$, and corresponding lowercase letters to denote the sides $(a, b, c, ...)$ opposite the angle. Figure 3.30 illustrates a right triangle labelled with Latin letters.

Considering the fact that we recently used properties of right triangles, and the Pythagorean theorem to solve for the x, and y coordinates along a circle of radius 1, then it may not come as a surprise that we can extend the use of trigonometric functions to encompass all right triangles regardless of the size of the hypotenuse. We'll begin by naming the sides of the triangle as opposed to simply referencing the legs as side x, or y; rather we'll refer to the sides with respect to the angle of interest which is known as the **angle of reference** (not to be confused with the *reference angle*). The following two figures illustrates this with θ, and γ being the angle of reference, and the side opposite the angle is abbreviated as **opp**, the side adjacent to the angle is abbreviated as **adj**, while the hypotenuse has the abbreviation **hyp**.

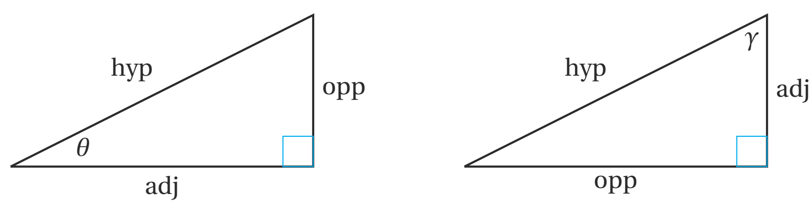


Figure 3.31

By labeling the triangle based upon the angle of reference, then the orientation of the right triangle is not relevant. For instance, if you study figure 3.32 you will see that the right triangle is simply rotated instead of being shown in *standard position*, yet the properties of trigonometry will still apply. With this labeling, the following six trigonometric functions are defined as the following:

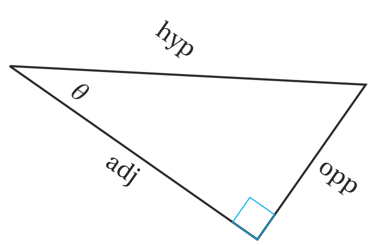


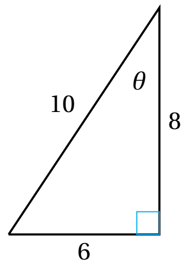
Figure 3.32

Trigonometric Functions

Sine of θ:	$\sin\theta = \dfrac{opp}{hyp}$	Cosecant of θ:	$\csc\theta = \dfrac{hyp}{opp}$
Cosine of θ:	$\cos\theta = \dfrac{adj}{hyp}$	Secant of θ:	$\sec\theta = \dfrac{hyp}{adj}$
Tangent of θ:	$\tan\theta = \dfrac{opp}{adj}$	Cosine of θ:	$\cot\theta = \dfrac{adj}{opp}$

Example 3.4.1: – Trigonometric values for complementary angle

Find the values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the following right triangle.



Solution:

First notice that θ is in the upper right corner of the triangle. The side with length 6 is opposite θ, while the other leg of length 8 is adjacent to θ.

$$\sin\theta = \frac{opp}{hyp} = \frac{6}{10} = \frac{4}{5}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{8}{10} = \frac{4}{5}$$

$$\tan\theta = \frac{opp}{adj} = \frac{6}{8} = \frac{3}{4}$$

Example 3.4.2: – Trigonometric function equal to a decimal value

If $\sin\alpha = 0.810$, then what are the values of the other five trigonometric functions?

Solution:

The best approach is to draw a right triangle with the information given. You'll most likely notice that $\sin\alpha = 0.810$ is not in fractional form as $\sin\alpha$ is defined; however, we could write 0.810 as either $\frac{810}{1000}$ as shown in figure 3.33, or as $\frac{0.810}{1}$ as shown in figure 3.34. It doesn't matter which way we choose as the result will be the same.

The Pythagorean theorem was used to find the values adjacent to α and was rounded to the thousandths place. For this reason, all of the following values are approximations.

$$\cos\alpha = \frac{586}{1000} = 0.586 \qquad \sin\alpha = \frac{810}{1000} = 0.810$$

$$\tan\alpha = \frac{810}{586} = 1.382 \qquad \cot\alpha = \frac{586}{810} = 0.723$$

$$\csc\alpha = \frac{1000}{810} = 1.235 \qquad \sec\alpha = \frac{1000}{586} = 1.706$$

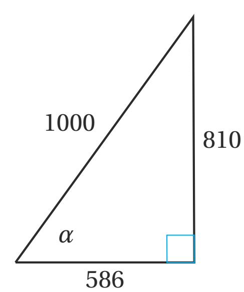


Figure 3.33

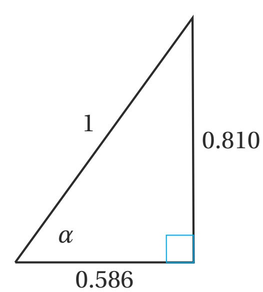


Figure 3.34

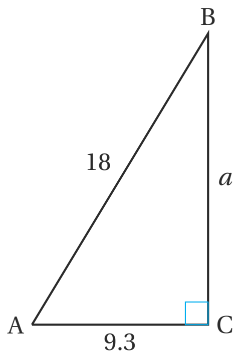


Figure 3.35

Example 3.4.3:

Solve the remaining parts of the right triangle ($\triangle ABC$) in figure 3.35.

Solution:

The following table shows what is known and what is needed to be found.

Sides	Angle
$a =$	$A =$
$b = 9.3$	$B =$
$c = 18$	$C = 90°$

We can determine side a from the Pythagorean theorem where $a^2 = c^2 - b^2 = (18)^2 - (9.3)^2 = 237.51$, thus $a = \sqrt{237.51} \approx 15.41$. Now that we have the lengths of all sides, we can determine either angle, $\angle A$ or $\angle B$ first. Beginning with $\angle A$, we could choose any of the inverse trigonometric functions to determine this angle, however we want to avoid using the side a we just found since its result was approximated, thus we'll use inverse cosine. Entering $\cos^{-1}\left(\frac{9.3}{18}\right)$ exactly as it is shown here in the calculator will give $A \approx 58.89°$. Since the sum of all angles in any triangle is 180°, then $B \approx 180° - 90° = 58.89° = 31.11°$. We can now complete the table shown below.

Sides	Angle
$a = 15.41$	$A = 58.89°$
$b = 9.3$	$B = 31.11°$
$c = 18$	$C = 90°$

In the above example 3.4.3 there are several options in the calculator to speed up calculations. For instance, we could have used the storing option to save the result of $\cos^{-1}\left(\frac{9.3}{18}\right) = 58.8911°$ as opposed to rounding the result and compounding the error. Another method is to use the arrow keys to scroll up to the result you want, or use the $\boxed{\text{ANS}}$ button/option to recall the previous answer. In any case, it's best to avoid rounding until the final result.

3.5 Applications of Trigonometry

3.5.1 Applications of Right Triangles

There are many applications for *right triangle trigonometry*, and with that there some terms you must be familiar with. The next example refers to a term called the *angle of depression*. The **angle of depression** is the angle that is produced when the object of interest is below the horizontal line created by the point of observation and the horizon where the point of observation represents the vertex. The **angle of elevation** is produced when the object of interest is above the horizontal line created by the point of observation and the horizon. Figure 3.36 illustrates both the angle of elevation and the angle of depression where the point of observation is represented by the persons viewpoint.

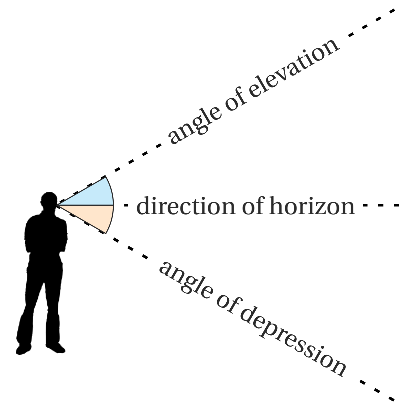
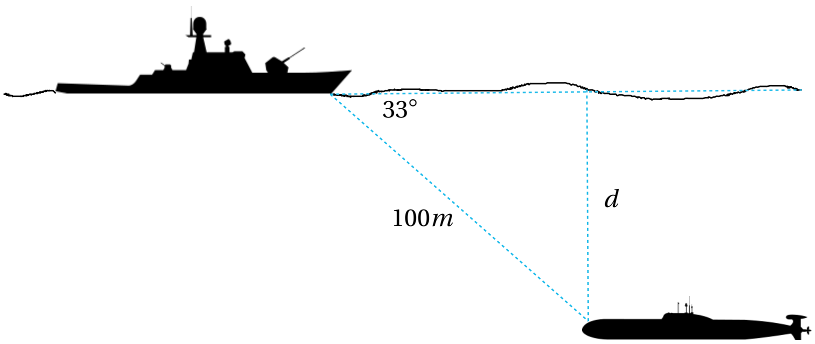


Figure 3.36

Example 3.5.1: –Angle of depression

A ships' radar has detected a submarine at a distance of $100m$ with an angle of depression of 33° as shown in the figure below. How far is the submarine from the surface of the water?



Solution:

In order to determine the submarines depth, we will have to make use of one of the trigonometric functions. Three of the trigonometric functions are reciprocals of the other three, so we can just restrict our choices down to sine, cosine, and tangent.

Since we're looking for d, then we must use sine since the information given deals with the side opposite 33°, and the hypotenuse.

$$\frac{opp}{hyp} = \sin(33^\circ)$$

$$\frac{d}{100} = \sin(33^\circ)$$

avoid approximating $\sin(33^\circ)$ until the end to avoid intermediate rounding errors.

$$\begin{aligned} d &= 100\sin(33^\circ) \\ &\approx 54 \end{aligned}$$

The submarine is approximately 54 meters from the surface.

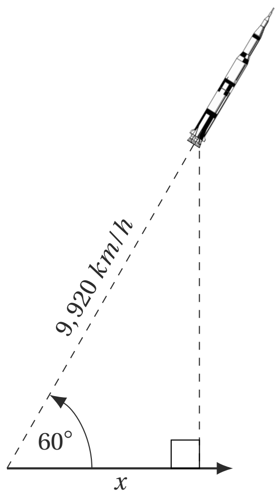


Figure 3.37

When dealing with applications involving moving objects we are primarily interested in the behavior of an object at a specific point in time or position. For instance, the next example asks us to determine the *ground speed* of a rocket at a specific point in time. **Ground speed** is the speed along the horizontal as described in section 1.2 on page 14.

Example 3.5.2: – Angle of elevation

The first stage of the Saturn V rocket burned for 2.5 minutes reaching a height of $68 km$ traveling at a speed of $9,920 km/h$ as illustrated in figure 3.37. If the Saturn V rocket was oriented at an angle of elevation with respect to the observers horizon, then what would the ground speed of the Saturn V be at the point of separation of the first stage rocket with respect to the observer?

Solution:

First, in some problems there may be some information that is not needed to solve for what is asked. In this case the height of the rocket at the moment of separation of the first stage is not relevant to determine the ground speed of the rocket. In addition, you may have already noticed that the hypotenuse of the triangle in the illustration is labeled as a rate instead of a constant which is okay as well. This just means that the result that we determine for horizontal will also be a rate which is what we want to determine. Now, we just need to choose the correct trigonometric function which gives us what we want based upon the information given. In this case $\cos 60°$ is the function we want since the horizontal value of x (adjacent side) is what we're looking for.

$$\frac{x}{9,920 \ km/h} = \cos 60°$$

$$x = 9,920 \cos(60°) \ km/h$$
$$= 4,960 \ km/h$$

Thus, the ground speed is 4,960 km/h

The above example was setup using the concept of *vectors*; however, the use of vectors is explained in much more detail in chapter **??**. The problem could also have been setup using the cosine function to determine the horizontal distance traveled then converted to the rate at that point. This method ultimately gives the same result with a little more intermediate work. The method in which we solved this problem is of course correct, but the diagram, using vectors, would have been portrayed slightly different which we'll discover in chapter **??**.

The next example revisits a problem from section 2.5 where we determined the height of a tower using similar triangles. Here we'll use trigonometry to solve for the height of the same tower where the angle of elevation and the length the shadow makes with the ground is known. In practice, the tool used to determine the angle of elevation, or depression, is called an *inclinometer* or *clinometer*. These types of tools range anywhere from a crude setup from a protractor, string, and plumb bob (figure 3.38) to digital high precision instruments.

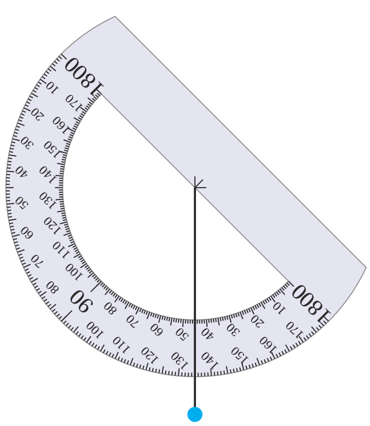


Figure 3.38: Protractor

Example 3.5.3:

The length of the shadow that the a tower makes along level ground is $389'2''$. The angle of elevation to the top of the tower from the end of the shadow is $31.45°$ which is shown in figure 3.39. What is the height of the tower to the nearest inch?

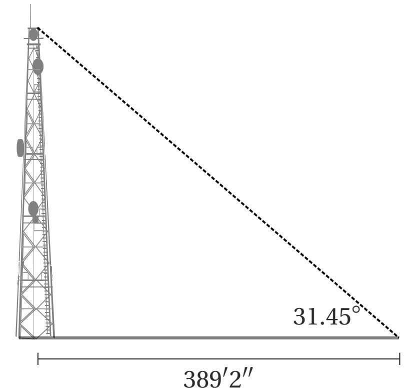


Figure 3.39

Solution:

In most cases it's helpful to redraw the diagram without all the unnecessary information such as the tower itself in this case, but since we are only looking for the height of the tower it's obvious what this diagram would look like based upon the illustration in figure 3.39.

We are looking for the height of the tower which is opposite our angle, and are given the length of the shadow the tower makes which is the adjacent side to the angle of elevation. Thus, we need to use the tangent function to determine the height.

$$\frac{h}{389+2/12} = \tan 31.45°$$

2/12 or 1/6 is 2" converted to feet, but is also a non-terminating decimal. To avoid rounding error we'll leave it as 1/6 till the final calculation.

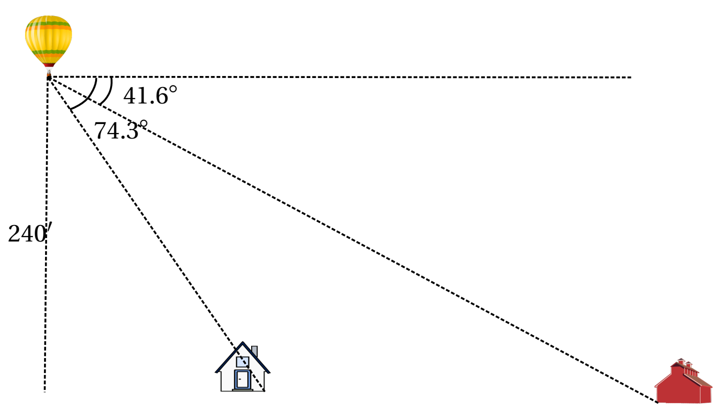
$$\begin{aligned} h &= (389+1/6)\tan 31.45° \\ &= 238.015' \\ &= 238.015(12) \\ &= 2{,}856.18'' \end{aligned}$$

Convert to inches

To the nearest inch, the tower is approximately 2,856 inches tall.

Example 3.5.4:

An observer from a hot air balloon with an altitude of 240' spotted a house and a barn. The angle of depression to the house is 74.3° while the angle of depression to the barn is 41.6°. What is the distance between the house and barn to the nearest foot?



Solution:

Begin by redrawing the diagram as shown in figure 3.40 and label what is known as well as what is needed. To find the distance from the house to the barn, we must find the horizontal distance of both the house and barn from the balloon and take their difference. To determine the horizontal distances in this case we'll have to use the tangent function since the horizontals are adjacent to the angles given.

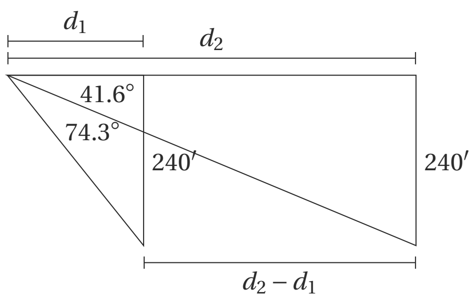


Figure 3.40

$$\tan 74.3^\circ = \frac{240}{d_1} \quad , \qquad \tan 41.6^\circ = \frac{240}{d_2}$$

$$d_1 = \frac{240}{\tan 74.3^\circ} \quad , \qquad d_2 = \frac{240}{\tan 41.6^\circ}$$

$$d_1 = 67.461' \quad , \qquad d_2 = 270.319'$$

Thus the distance between the house and barn is $d_2 - d_1 = 67.461' - 270.319' = 203'$ rounded to the nearest foot.

4-to-1 Rule:
In construction, the 4-to-1 rule states that for every 4 vertical feet a ladder is placed against a wall the base of the ladder should be placed one foot from the wall. For instance, if a ladder reaches 12 feet up a wall, then the base of the ladder should be placed at three feet from the wall.
If the base of the ladder is too close to the wall then the there is a risk for the top of the ladder to pull away from the wall as it is being used. If the base of the ladder is too far from the wall then there is risk the base will slide further out from the wall as it's being used. In either case, there is risk for injury if a ladder is not properly set.

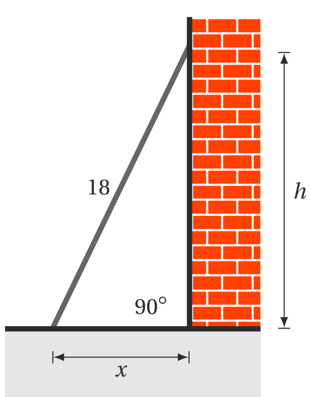


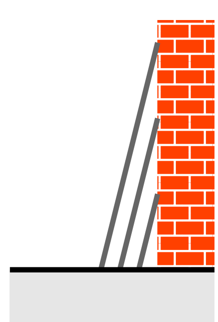
Figure 3.41

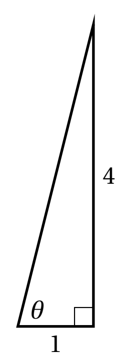
Example 3.5.5:

How far should the base of an 18 foot ladder be placed from a wall based on the 4-to-1 rule?

Solution:

First, notice the illustration in figure 3.41 for the ladder/wall relationship. If we look closely enough, you'll notice that there doesn't seem to be enough information in the diagram to determine the value for x since the vertical height h wasn't given. Rather, all we have is the length of the ladder with the other two legs of the right triangle unknown. To solve the remaining parts of any triangle, there *must* be at least three pieces of information known about the triangle. In this case, we only know two; however, we do have the details of the 4-to-1 rule. If you study the illustration below of several ladders following the 4-to-1 rule then you'll notice that the angle the ladder makes with the ground is the same since they all form similar triangles.





For this reason, we can determine the angle of elevation the ladder makes with respect to the ground using the 4-to-1 rule since all the angles would be the same which is illustrated as a right triangle in the figure above.

$$\tan\theta = \frac{4}{1}$$
$$\theta = \tan^{-1}(4)$$
$$\approx 75.96^\circ$$

Since we are looking for the adjacent side, we'll use the cosine function to solve for x when the hypotenuse is 18.

$$\frac{x}{18} = \cos 75.96^\circ$$
$$x = 18\cos 75.96^\circ$$
$$\approx 4.3668'$$

The base of an 18' ladder should be placed approximately 4'4" from the wall.

Example 3.5.6: – Solving Trigonometric Equations

Two radar stations located 10 km apart both detect a UFO located between them. The angle of elevation measured by the first station (A) is 36° and the angle of elevation measured by the second station (C) is 20°. What is the altitude (h) of the UFO? See figure 3.43 below.

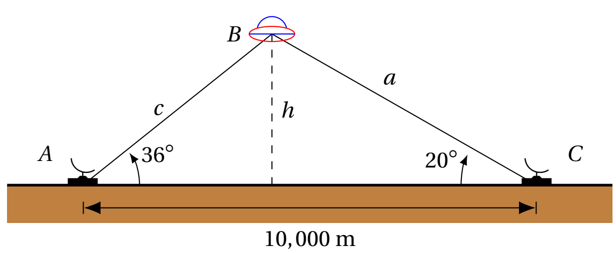


Figure 3.43: UFO and radar stations

Solution: To begin, we need to redraw a digram that allows us to determine two right triangles where the distance between the radar station A and the UFO will be denoted as x, and the horizontal distance between radar station B will be denoted as $10{,}000 - x$ (see figure 3.42). With this new labeling we can setup two right triangles with common variables. In each of the following expressions both x and h refer to the same values.

$$\tan 36^\circ = \frac{h}{x} \qquad and \qquad \tan 20^\circ = \frac{h}{10{,}000 - x}$$
$$h = x\tan 36^\circ \qquad and \qquad h = (10{,}000 - x)\tan 20^\circ$$

Since both expressions are equal to h, then the expressions are equal to each other. This will leave us with one equation where we must solve for x.

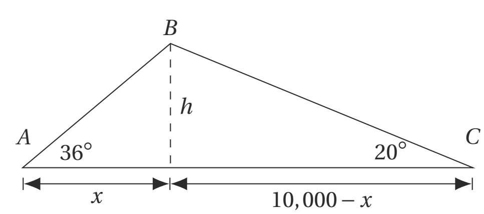


Figure 3.42

$$x\tan 36^\circ = (10,000 - x)\tan 20^\circ$$

$$x\tan 36^\circ = 10,000\tan 20^\circ - x\tan 20^\circ$$ distribute $\tan 20^\circ$ throughout $(10,000 - x)$.

$$x\tan 36^\circ + x\tan 20^\circ = 10,000\tan 20^\circ$$ add $x\tan 20^\circ$ to both sides

$$x(\tan 36^\circ + \tan 20^\circ) = 10,000\tan 20^\circ$$ factor x out of the right side of the equation

$$x = \frac{10,000\tan 20^\circ}{\tan 36^\circ + \tan 20^\circ}$$

$$\approx 3337.61m$$

Now that we have x we can substitute its value into either expression above to determine h. Choosing the simplest expression we have $h = x\tan 36^\circ = 3337.61\tan 36^\circ = 2424.91m$. Therefore the height of the UFO is approximately 2425 meters.

3.5.2 Applications with Circular Motion

One application of radian measure occurs when we deal with velocity of an object traveling in a circular motion, or simply rotational motion. To review radian measure see sections 2.1.3 and 3.1.2. When determining the distance of an object traveling in a straight line, we use the formula $d = vt$ which is distance equals velocity times time. When determining the distance of an object travelling in a circular motion, recall that the arc length is defined as $s = r\theta$ where θ is in radians, r is the radius of the arc, and s is the distance around the arc. If we let v represent the *velocity* or rate, then $s = vt$ which is also distance equals velocity times time. After substituting $s = r\theta$ we have $r\theta = vt$. From this equation we can solve for any unknown variable. For example, solving for velocity we get $v = \dfrac{r\theta}{t}$.

Example 3.5.7: – Linear Velocity

A wheel with a radius of 20” spins at the rate of 7 revolutions every 1.03 seconds. What is the velocity of the wheel?

Solution:

As shown above, $v = \dfrac{r\theta}{t}$. We know $t = 1.03s$, r=20”, and $\theta = 2\pi(7) = 14\pi$ we have the following:

$$v = \frac{r\theta}{t}$$

$$v = \frac{20(14\pi)in}{1.03s}$$

$$= 854.025\ in/s$$

Example 3.5.8: – Linear Velocity

The radius of the earth is 3960 miles and makes one revolution every 24 hours approximately. What is the velocity at a point along the equator in miles per hour?

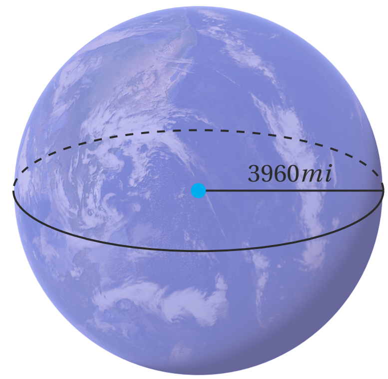


Figure 3.44: Radius of Earth

Solution:

Since we are looking for the velocity along the equator then we can continue to use the radius of the earth.

$$v = \frac{r\theta}{t}$$

$$v = \frac{3960(2\pi)mi}{24h}$$

$$= 1036.73\ mi/h$$

A spot on the equator has as a velocity of 1036.73 mi/h.

Recall from earlier that the velocity of an object moving in circular motion is given by $v = \frac{r\theta}{t}$, which is equivalent to $v = r\left(\frac{\theta}{t}\right)$. The ratio of $\frac{\theta}{t}$ is what's called **angular velocity**, denoted with the lowercase Greek letter omega (ω), is the rate at which an object rotates through an angle θ in time t. Thus we have $\omega = \frac{\theta}{t}$. **Linear velocity** defined earlier in this section as $v = \frac{r\theta}{t}$ can also be written as $v = r\omega$.

Example 3.5.9: – Angular Velocity

A certain type of aluminum requires a cut speed of 40 feet per minute to prevent damaging both the material, and cut tool. If round aluminum stock is to be turned on a machining lathe, and has a diameter of 6 inches, at what rpm should the lathe be set to rotate?

Solution:

We know $v = r\omega$, and need to solve for the angular velocity ω which is $\omega = \frac{v}{r}$. In the past, the units of measurement have been omitted from most calculations, but when multiple units of measurement are used it is not recommended to omit them.

$$\omega = \frac{v}{r}$$

$$= \frac{40\ ft/min}{3in}$$

notice that we need to either convert 3" to feet, or 40' to inches.

$$= \frac{40\ ft/min}{0.25ft}$$

$$\omega = 160 \frac{\cancel{ft}}{\cancel{ft} \cdot min}$$

Simplify the units and cancel like units.

$$= 160 \, \frac{rad}{min}$$

ω is in radians by default though it can be converted to degrees if needed at this point. We don't want to convert to degrees in this case.

Now, we just need to determine the number of revolutions per minute. Since 1 revolution is equal to 2π radians, we have the following:

$$\begin{aligned}
\omega &= 160 \, \frac{rad}{min} \\
&= 160 \, \frac{\cancel{rad}}{min} \cdot \frac{1 \text{ rev}}{2\pi \, \cancel{rad}} \\
&= \frac{160 \text{ rev}}{2\pi \, min} \\
&\approx 25.46 \text{ rpm}
\end{aligned}$$

Therefore the lathe needs to be set at 25.5 rotations per minute (rpm).

Chapter 4

Vectors and Oblique Triangles

4.1 Vectors in the Plane

4.1.1 Introduction

We deal with many quantities that are represented by a number that shows their magnitude. These include speed, money, time, length and temperature. Quantities that are represented only by their **magnitude** or size are called **scalars** . When you travel in your car and you look at the speedometer it tells you how fast you are going but not where you are going. This is a scalar value and is called the **speed**.

A **vector** is a quantity that has both a *magnitude* (size) and a *direction.* To describe a vector you must have both parts. If you know that you are traveling at 150 mph north then that would be a vector quantity and it is called the **velocity**. It tells you how fast you are traveling, speed is 150 mph, as well as the direction, north.

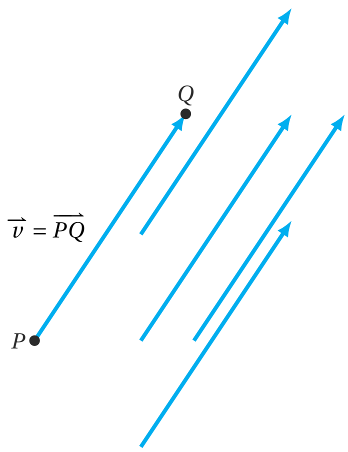


Figure 4.1: Equivalent Vectors

4.1.2 Vector Representation

When we write a vector there are two common ways to do it. If we want to talk about “vector v” we can either write the **v** in bold or write the $\vec{v}$ with an arrow over it. In this text we will most often use the arrow notation but do be aware that the bold notation is also common.

To describe a vector we need to talk about both the magnitude and direction. The magnitude of a vector is represented by the notation $\|\vec{v}\|$. The direction can be described in different ways and depends on the application. For example you might say that a jet is traveling in the direction 10° north of east, or a force is applied at a particular angle or with a particular slope. A vector can be represented by simply an arrow: in Figure 4.1 the vector $\vec{v} = \overrightarrow{PQ}$ which starts at point P called the **initial point**, and ends at point Q called the **terminal point** has magnitude equal to its length ($\|\vec{v}\|$) and direction as indicated. The vector can be moved around in the plane as long as the length and direction are unchanged. All the vectors in Figure 4.1 are equivalent because they all have the same length and point in the same direction. When the vector is drawn this way the length is always the magnitude. An accurate picture is necessary to accurately describe a vector this way. Sometimes it is called a **directed line segment**.

Example 4.1.1:

Show that the directed segment $\vec{u}$ which starts at $P(-3,-2)$ and ends at $Q(1,4)$ is equivalent to the directed segment $\vec{v}$ which starts at $R(3,1)$ and ends at $S(7,7)$.

Solution:

To show that the two vectors are equivalent we need to show that they have the same length and direction as illustrated in Figure 4.2. Using the distance formula we can see they have the same length.

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(1-(-3))^2+(4-(-2))^2}\\ &= \sqrt{4^2+6^2}\\ &= 2\sqrt{13}\end{aligned}$$

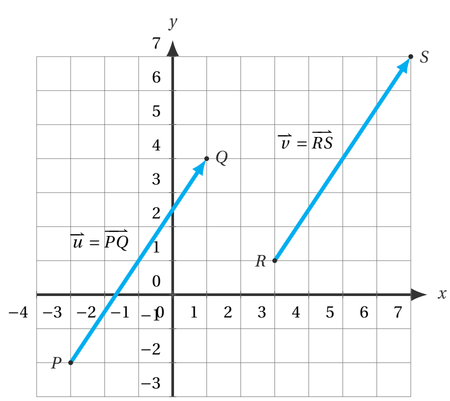


Figure 4.2

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(7-3)^2+(7-1)^2} \\ &= \sqrt{4^2+6^2} \\ &= 2\sqrt{13}\end{aligned}$$

Both of these vectors have the same direction because they are both pointing to the upper right and have the same slope:

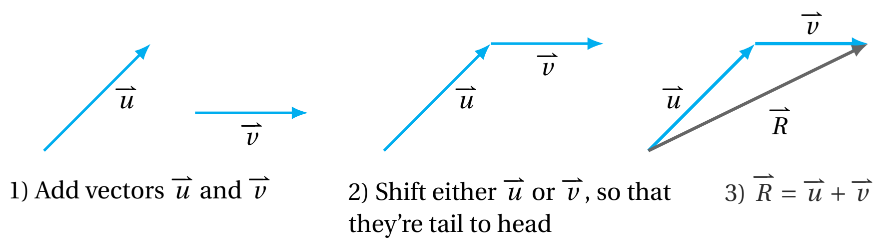
$$\frac{\Delta y}{\Delta x} = \frac{(4-(-2))}{(1-(-3))} = \frac{(7-3)}{(7-1)} = \frac{3}{2}$$

Thus they are equivalent.

4.1.3 Graphical Addition of Vectors

Polygon Method ▸

There are two common methods to adding vectors graphically. The first is known as the **polygon method**. To keep things simple, we'll begin with the addition of two vectors. The *polygon method* shifts either of the two vectors such that the initial point of one vector is at the same location as the terminal point of the other as illustrated in figure below. It's important to note that both the direction and magnitude is unchanged in this relocation step. The vector sum, say $\vec{u}+\vec{v}$, is typically denoted as $\vec{R}$, or in boldface **R**, which is called the **resultant vector**, or just **resultant**; thus we have $\vec{r}=\vec{u}+\vec{v}$.



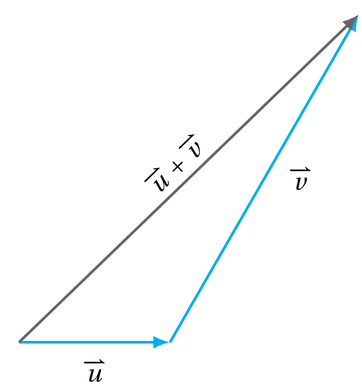


Figure 4.3

Figures 4.4 illustrates the *commutative property* of vector addition. Since the location of the vectors is not relevant, then regardless which vectors initial point is shifted to the terminal point of the other vector the resultant vector is the same.

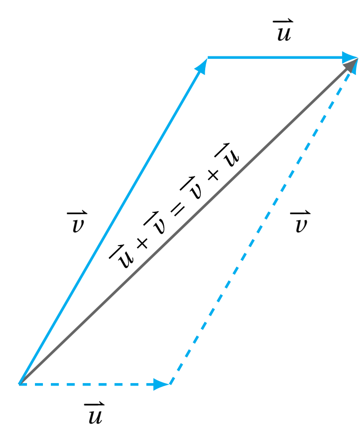
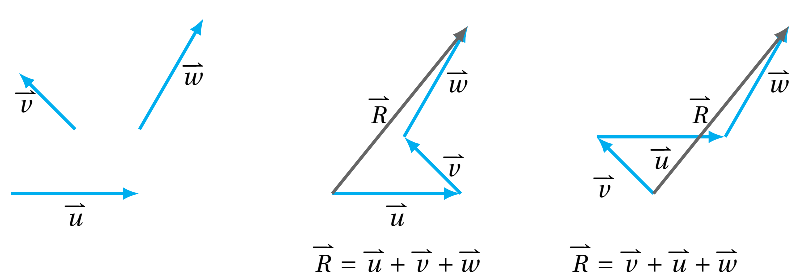


Figure 4.4

When adding three or more vectors as illustrated below in Example 4.1.2, the process is done in a similar manner. Each vectors initial point is placed at the terminal point of the previous vector, and the resultant vector begins at the initial point of the first vector while the terminal point is the terminal point of the last vector shifted. The order of the vectors shifted does not matter.

Example 4.1.2: –Vector addition - polygon method



Notice in Example 4.1.2 that the magnitude and direction of the resultant vector is unchanged regardless of the order in which the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are added.

Parallelogram Method ▸

The other method, which is typically convenient when adding only two vectors at a time, is called the **parallelogram method**. This method requires the position of two vectors such that they share the same initial point, thus the vectors become two sides of a parallelogram. The resultant is the diagonal of the parallelogram that shares the same initial point as illustrated in Figure 4.1.3 below.

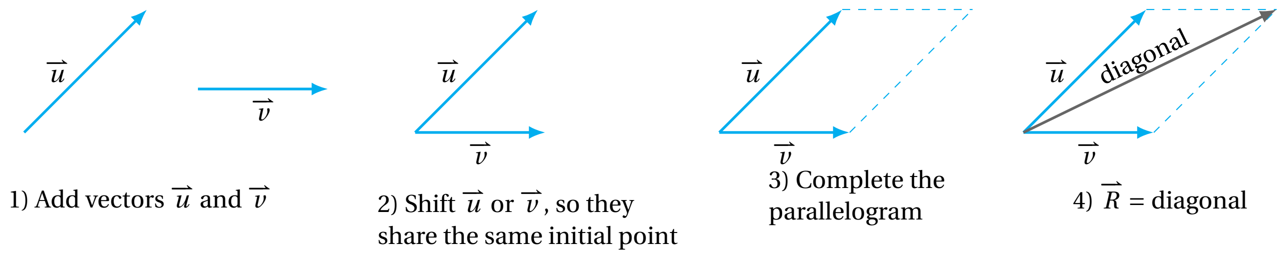


Figure: 4.1.3

4.1.4 Difference of two Vectors

Recall from section 4.1 that a negative vector simply changes the direction of the vector by exactly 180°. To take the difference between two vectors, say $\vec{u} - \vec{v}$, we changed the direction of $\vec{v}$ by adding 180° to its existing angle. Graphically, we simply drew the arrow on the other side of the vector, placed them tail to head, then drew a vector beginning at the initial point of $\vec{u}$ and ending at the terminal point of $\overrightarrow{-v}$ as illustrated in the step-by-step graphical approach in Figure 4.5 below.

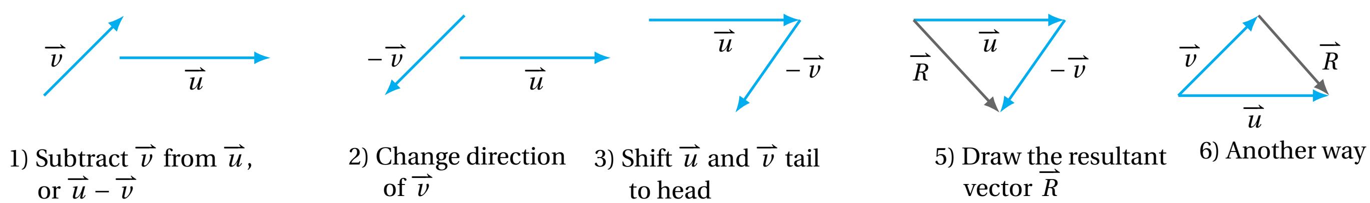


Figure 4.5: Difference of two vectors

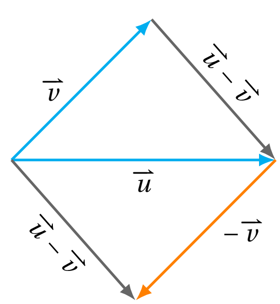


Figure 4.6: Difference of vectors in standard position

Notice in step 6 labelled "*Another way*" that the resultant vector appears to have both the same magnitude and direction as the resultant in stop 5), yet the vectors in step 6 aren't drawn tail-to-end. This is not a coincidence, rather studying Figure 4.6 will help convince you. Thus, the law of cosines can be used to determine the magnitude of the difference of two vectors in standard position since the angle between them at the origin is either given or relatively easy to determine. However, care should be taken when determining the direction of the resultant vector when calculated this way (see Figure 4.7)

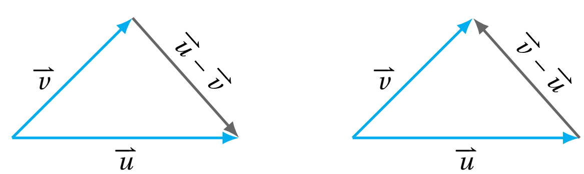
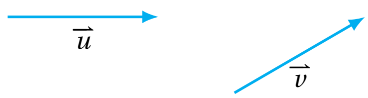


Figure 4.7

Example 4.1.3:

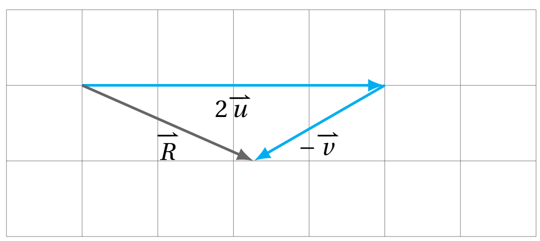
For the following vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ find $2\overrightarrow{u} - \overrightarrow{v}$.



Solution:

Since we are subtracting, we can redraw the vectors with the direction of $\overrightarrow{v}$ reversed.

Now, we just add them together by placing them tail-to-head as before.

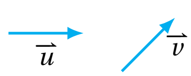


4.1.5 Scalar Multiple of Vectors - Graphically

If vector $\overrightarrow{u}$ is in the same direction as vector $\overrightarrow{v}$ where $\overrightarrow{u}$ has a magnitude n times that of $\overrightarrow{v}$, then $\overrightarrow{u} = n \cdot \overrightarrow{v}$, and the vector $n\overrightarrow{v}$ is called the **scalar multiple** of vector $\overrightarrow{v}$. In other words, a *scalar multiple* is a constant multiple that stretches or contracts a vector by a scale factor of n (see Figure 4.8). In the instance that n is negative the direction of the vector is reversed, or rotated 180°.

Example 4.1.4: – Scalar multiple

For the following vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ find $2\overrightarrow{u} + 3\overrightarrow{v}$.



Solution:

As stated before, adding vectors graphically requires precision in their placement. This is especially true when adding scalar multiples of vectors. The use of graph paper will help. Now, just redraw the vectors end-to-end, but with their respective scalar multiples.

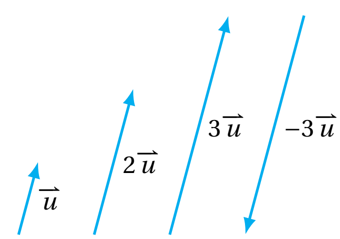
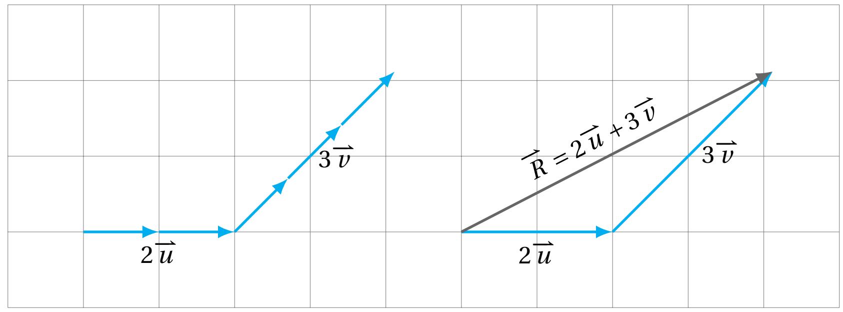


Figure 4.8

It's not necessary to draw the vectors multiple times. In Example 4.1.4 it was shown for explanation purposes; however, though not necessary it sometimes helpful.



4.2 Vector Components

4.2.1 Components of a Vector

Up to this point all the vectors introduced have been defined graphically as a directed line segment. Using diagrams to illustrate, and combine vectors is very useful when developing an understanding of vectors, but it's not practical when accuracy is needed. In this section we look at other methods to achieve much more accurate results when combining vectors.

A vector drawn starting at the origin is in **standard position** as shown in Figure 4.9. A vector in standard position has initial point at the origin $(0,0)$ and can be represented by the endpoint of the vector (a,b). This is known as representing the **vector by components**: $\overrightarrow{v} = \langle a, b\rangle$. It is common to see this written as $\overrightarrow{v} = \langle v_x, v_y\rangle$. See Figure 4.9. Notice the use of "angle brackets" $\langle\ \rangle$ to write the vector. This distinguishes it from the point at the end of the vector. Writing a vector as components is generally preferable because it is easier to perform calculations with components rather than directed line segments. Also, while all the work in this book is with two dimensional vectors you can also write vectors in three or even more dimensions. It is very difficult to draw a directed segment in three dimensions while writing it with components is quite straight forward.

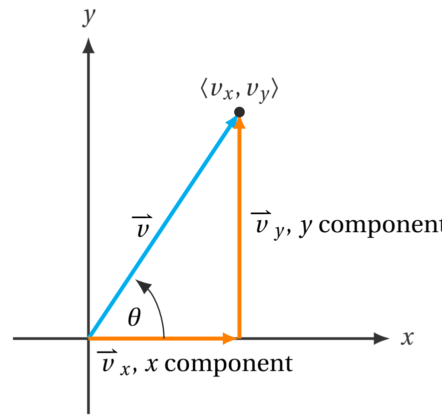


Figure 4.9: Vector Components

Note:
The x-component and y-component are sometimes referred to as the *horizontal component* and the *vertical component* respectively.

If you want to write $\overrightarrow{v}$ from point P to point Q then $\overrightarrow{v} = P - Q$. For example in **Example** 4.1.1 $\overrightarrow{u} = \overrightarrow{PQ} = (1,4) - (-3,-2) = \langle 4,6\rangle$. It is important to subtract in the correct order. It is always "end point" minus "starting point". If you subtract in the wrong order you end up with a vector that has the same length but points in the opposite direction.

Component Form of a Vector

The component form of a vector $\overrightarrow{v}$ with initial point $P(p_1,p_2)$ and end point $Q(q_1,q_2)$ is

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2\rangle = \langle v_x, v_y\rangle = \overrightarrow{v}$$

The magnitude of $\overrightarrow{v}$, $\|\overrightarrow{v}\|$, is found by the Pythagorean theorem.

$$\|\overrightarrow{v}\| = \sqrt{(q_1-p_1)^2 + (q_2-p_2)^2} = \sqrt{(v_x)^2 + (v_y)^2}$$

Example 4.2.1:

Find the component form of the vector $\overrightarrow{v}$ that starts at $P(1,2)$ and ends at $Q(-3,4)$. Find the length of $\overrightarrow{v}$.

Solution:

$$\begin{aligned}\overrightarrow{v} &= \langle q_1 - p_1, q_2 - p_2\rangle \\ &= \langle (-3-1), (4-2)\rangle \\ &= \langle -4, 2\rangle\end{aligned}$$

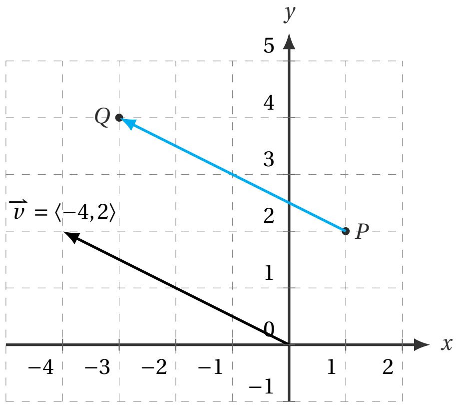


Figure 4.10

The length:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(v_x)^2 + (v_y)^2}\\ &= \sqrt{(-4)^2 + (2)^2}\\ &= \sqrt{20} = \sqrt{4}\sqrt{5}\\ &= 2\sqrt{5}\end{aligned}$$

See Figure 4.10 for an illustration of this problem.

4.2.2 Vector Operations - by Components

There are mathematical operations that we can do with vectors. The two most common are **multiplication by a scalar** and **vector addition** . Recall that a scalar is a number. If you want to multiply a vector $\vec{v}$ by a scalar n there are two ways to think about it. Multiplying by the scalar n does not change the direction of the vector but makes it longer or shorter by a factor of n. If you have your vector written in components $\vec{v} = \langle v_x, v_y\rangle$ then each component is multiplied by n:

$$n \cdot \vec{v} = n \cdot \langle v_x, v_y\rangle = \langle n \cdot v_x, n \cdot v_y\rangle$$

Example 4.2.2: – Scalar multiple of Vector Components

Find the result when $\vec{u} = \langle 6, -1\rangle$ is multiplied by 7.

Solution:

$$7\vec{u} = 7\langle 6, -1\rangle = \langle 42, -7\rangle$$

Example 4.2.3: – Scalar multiple of Vector Components

Find the result when $\vec{u} = \langle 6, -1\rangle$ is multiplied by -1.

Solution:

$$(-1)\vec{u} = -\vec{u} = (-1)\langle 6, -1\rangle = \langle -6, 1\rangle$$

Note that $-\vec{u}$ is the same vector as $\vec{u}$ but pointing in the opposite direction. You can see this if you sketch both on the same set of axes.

If the vectors are written as components you can add the x components and the y components separately. The component operations are summarized below.

Vector Addition and Scalar Multiplication

Given vectors $\overrightarrow{u} = \langle u_x, u_y\rangle$ and $\overrightarrow{v} = \langle v_x, v_y\rangle$ and scalar n then the sum or difference of $\overrightarrow{u}$ and $\overrightarrow{v}$ is given by

$$\overrightarrow{u} + \overrightarrow{v} = \langle u_x, u_y\rangle + \langle v_x, v_y\rangle = \langle u_x + v_x, u_y + v_y\rangle$$

$$\overrightarrow{u} - \overrightarrow{v} = \langle u_x, u_y\rangle - \langle v_x, v_y\rangle = \langle u_x - v_x, u_y - v_y\rangle$$

The scalar multiple of k and $\overrightarrow{v}$ is

$$n \cdot \overrightarrow{v} = n \cdot \langle v_x, v_y\rangle = \langle n \cdot v_x, n \cdot v_y\rangle$$

Example 4.2.4: –Vector Addition and Scalar Multiplication by Components

Let $\overrightarrow{u} = \langle 1, -2\rangle$ and $\overrightarrow{v} = \langle -4, 2\rangle$, and find

a) $3\overrightarrow{u} + \overrightarrow{v}$

b) $\overrightarrow{u} - \overrightarrow{v}$

c) $\overrightarrow{v} - 2\overrightarrow{u}$

Solution:

To add these we need to add the corresponding components. The order of operations is still valid here, perform the scalar multiplication first and then the vector addition.

a) $3\overrightarrow{u} + \overrightarrow{v} = 3\langle 1, -2\rangle + \langle -4, 2\rangle = \langle 3, -6\rangle + \langle -4, 2\rangle = \langle -1, -4\rangle$

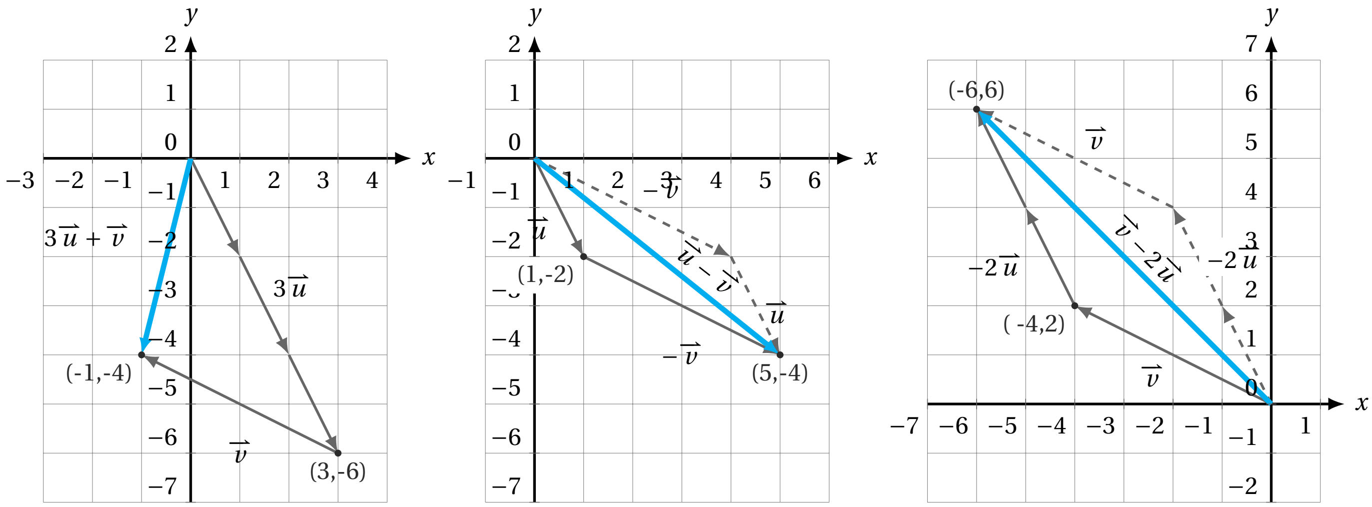
   The solution is also shown in Figure 4.11 (a)

b) $\overrightarrow{u} - \overrightarrow{v} = \langle 1, -2\rangle - \langle -4, 2\rangle = \langle 5, -4\rangle$

   To do this with arrows on paper it is easiest to draw $-\overrightarrow{v}$ and then add that to $\overrightarrow{u}$. Remember that $-\overrightarrow{v}$ is the same as $\overrightarrow{v}$ but the arrow is on the other end of the vector. The solution is shown in Figure 4.11 (b) Notice that we can add in either order, the dotted vectors are the result of $-\overrightarrow{v} + \overrightarrow{u}$

c) $\overrightarrow{v} - 2\overrightarrow{u} = \langle -4, 2\rangle - 2\langle 1, -2\rangle = \langle -4, 2\rangle + \langle -2, 4\rangle = \langle -6, 6\rangle$

   Be careful with the sign when multiplying by the -2. The solution is shown in Figure 4.11 (c). Notice that we can add in either order, the dotted vectors are the result of $-2\overrightarrow{u} + \overrightarrow{v}$



a) $3\vec{u} + \vec{v}$ b) $\vec{u} - \vec{v}$ c) $\vec{v} - 2\vec{u}$

Figure 4.11: Illustrations for Example 4.2.4

4.2.3 Trigonometric Derived Vector Components

In application, many vectors are given in the form of a direction and a scalar quantity which we depict to be represented as the magnitude. Thus, the horizontal and vertical components of the vector are not usually given. Nevertheless, the horizontal and vertical components can be determined given the direction and magnitude of a vector. The initial points of these vector components are always located at the origin as illustrated in Figure 4.12.

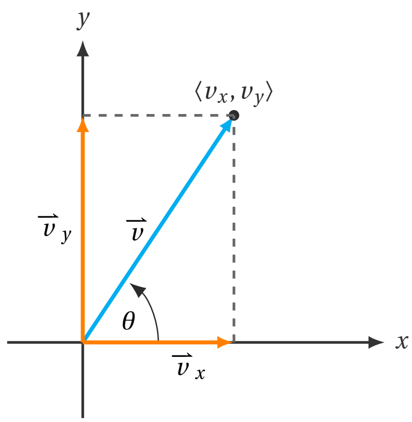


Figure 4.12: Vector Components

When we stop to consider that a vector drawn in standard position (initial point at the origin) is simply a directed line segment, then it's not too difficult to visualize that, nearly, any vector can represent the hypotenuse of a triangle. Thus the magnitude of the x, or horizontal, component is determined by $\|\vec{v}_x\| = \|\vec{v}\| \cdot \cos\theta$, and the magnitude of the y, or vertical, component is determined by $\|\vec{v}_y\| = \|\vec{v}\| \cdot \sin\theta$ which is shown below. The one exception to this *triangular* visualization is if the direction of the vector is a multiple of 90° which doesn't create a triangle; however, the expressions just described still provides the correct values for the components in those cases.

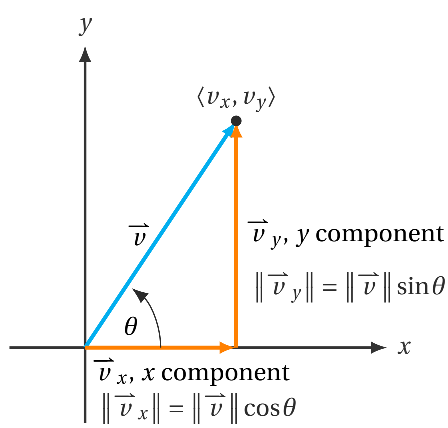


Figure 4.13

Example 4.2.5: – Components in the First Quadrant

Find the x- and y- components of the vector with magnitude 10 in the direction of 30°

Solution:

x component:

$$\begin{aligned}\|\vec{v}_x\| &= \|\vec{v}\|\cos\theta\\ &= 10\cos 30^\circ\\ &= 10\left(\frac{\sqrt{3}}{2}\right)\\ &= 5\sqrt{3}\end{aligned}$$

y component:

$$\begin{aligned}\|\vec{v}_y\| &= \|\vec{v}\|\sin\theta\\ &= 10\sin 30^\circ\\ &= 10\left(\frac{1}{2}\right)\\ &= 5\end{aligned}$$

The vector with its components are shown in Figure 4.14

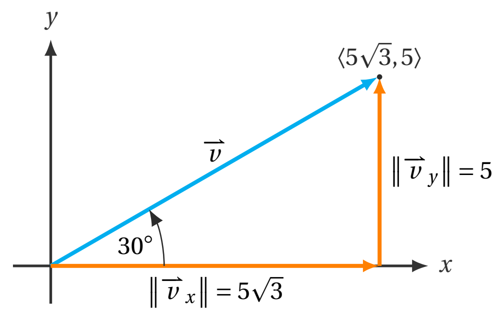


Figure 4.14

Care and attention should be taken when working with vectors located in other quadrants. For instance, if a vector is in the quadrant II then we would expect the x component to be negative, and the y component to be positive, while vectors in quadrant IV have signs that are the converse of quadrant II. Vectors in quadrant III are expected to have both x and y components negative. It is always a good strategy to draw a diagram of the problem; not just to better understand the problem, but the results found can be compared so that they make sense.

Example 4.2.6:

Find the components of vector $\vec{u}$ with magnitude 162 and direction $\theta = 203.37^\circ$.

Solution:

Begin by drawing a diagram of the vector described (shown in Figure 4.15). This will help ensure that our answers make sense. There are two ways to approach this type of problem where the first is to use the angle given at 203.37° (preferable in this case), and the second is to use the reference angle of $203.37^\circ - 180^\circ = 23.37^\circ$. If we use the reference angle the resultant components will be in the first quadrant and their signs would have to be changed to negative to place them in the third quadrant.

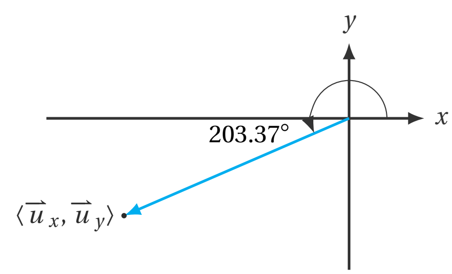


Figure 4.15

Note:
The vector labeled $\langle \vec{u}_x, \vec{u}_y\rangle$ is equivalent to $\langle \|\vec{u}_x\|, \|\vec{u}_y\|\rangle$.

x- component:

$$\begin{aligned}\|\vec{u}_x\| &= \|\vec{u}\|\cos\theta\\ &= 162\cos 203.37^\circ\\ &= -148.71\end{aligned}$$

y- component:

$$\begin{aligned}\|\vec{u}_y\| &= \|\vec{u}\|\sin\theta\\ &= 162\sin 203.37^\circ\\ &= -64.26\end{aligned}$$

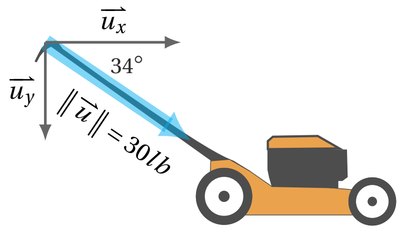


Figure 4.16

Example 4.2.7: – Vector Components

A lawnmower is pushed with a force of 30 pounds with an angle of depression of 34° as shown in Figure 4.16. What are the horizontal and vertical forces acting on the lawnmower?

Solution:

x- component:

$$\begin{aligned}\overrightarrow{u}_x &= \|\overrightarrow{u}\|\cos\theta\\ &= 30\cos(-34°)\\ &= 24.8711\ lb\end{aligned}$$

y- component:

$$\begin{aligned}\overrightarrow{u}_y &= \|\overrightarrow{u}\|\sin\theta\\ &= 30\sin(34°)\\ &= -16.7758\ lb\end{aligned}$$

Notice the vertical component, $\overrightarrow{u}_y$, is negative. The reason for this is that I had relocated the vector in the Cartesian plane with the initial point of $\overrightarrow{u}$ located at the origin. This placement makes the angle of depression to be either −34°, or its coterminal equivalent 326°. Thus, the vertical component would be negative. A negative component only means that the direction is either down/backward, while a positive component is up/forward. In this case, the force moving the lawnmower forward is approximately 25 lb, and there is approximately 17 lb of force directed toward the ground.

Note:
See Appendix A for information on directional heading, bearing, and the difference between them.

Example 4.2.8: – Combining Vectors

To avoid a storm a jet travels $N70°E$, or equivalently 20° north of east, from Winnipeg for 300 km and then turns to a heading $S28°E$, or equivalently 62° south of east, for 1150 km to arrive at Chicago. Find the displacement from Winnipeg to Chicago.

Solution:

Figure 4.17 shows the flight path. It is a good idea to draw a picture if possible. While it would be possible to try and measure the vectors and angles it will be easier to add these by components. We will calculate the components for each leg of the journey and then add them up. For the first leg $\overrightarrow{L1} = \langle L1_x, L1_y\rangle$ we have

$$L1_x = 300\cos(20°) = 282$$

$$L1_y = 300\sin(20°) = 103$$

For the second leg $\overrightarrow{L2} = \langle L2_x, L2_y\rangle$ we have

$$L2_x = 1150\cos(62°) = 540$$

$$L2_y = -1150\sin(62°) = -1015$$

It is important to notice that the y component is negative because it points in the negative y direction. The picture will help make sure the signs are correct on your components.

The resultant vector is $\overrightarrow{L1} + \overrightarrow{L2} = \langle 282, 103\rangle + \langle 540, -1015\rangle = \langle 822, -912\rangle$. The distance from Winnipeg to Chicago is the magnitude of the resultant vector. So the displacement is $\sqrt{822^2 + (-912)^2} = 1228$ km.

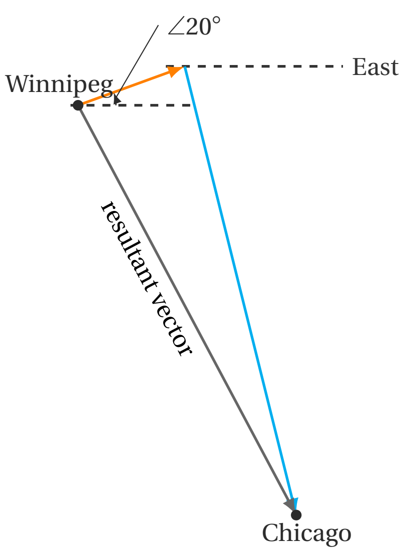


Figure 4.17

Example 4.2.9: – Combining Vectors

An airplane is on a heading of $N37°W$ with a ground speed of 750 km/hr when it encounters a strong wind with a velocity 100 km/hr at a bearing of $N30°E$. Find the resultant speed and direction (bearing) of the airplane. Figure 4.18

Solution:

The resultant speed and direction of the airplane ($\vec{R}$) is the sum of the plane's ground speed velocity vector and the wind speed vector. Figure 4.18 shows the relationship between the vectors. To add them we will first write them as components. Let $\vec{P} = \langle P_x, P_y \rangle$ be the airplane ground speed vector and $\vec{W} = \langle W_x, W_y \rangle$ be the wind speed vector.

$$\begin{aligned}\vec{P} &= 750\langle\cos(127°), \sin(127°)\rangle \\ &\approx \langle -451, 599\rangle \text{ km/hr} \\ \vec{W} &= 100\langle\cos(60°), \sin(60°)\rangle \\ &\approx \langle 50, 87\rangle \text{ km/hr}\end{aligned}$$

Note the signs on the components of the vectors and compare them to the figure. You expect the x component of the airplane's ground speed vector to be negative, and it is.

So the velocity of the plane in the wind is

$$\begin{aligned}\vec{R} &= \vec{P} + \vec{W} \\ &\approx \langle -451, 599\rangle + \langle 50, 87\rangle \\ &\approx \langle -401, 686\rangle \text{ km/hr}\end{aligned}$$

and the resultant speed of the airplane

$$\begin{aligned}||\vec{R}|| &\approx \sqrt{(-401)^2 + (686)^2} \\ &\approx 795 \text{km/hr}\end{aligned}$$

For the bearing we will use the angle θ made with the negative x axis as shown in the figure.

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{686}{401}\right) \\ &\approx 59.7°\end{aligned}$$

which we write as bearing 329.7°, or $N30.3°W$. And we can put them together to say the airplane is traveling at 795 km/hr bearing 329.7°.

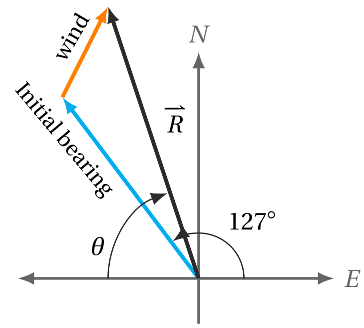


Figure 4.18

4.3 Applications of Vectors

4.3.1 Applications of Vectors

In this section we apply the principles from the previous two sections. There are many applications for vectors spanning from physical applications, opticals, data structures, chemistry, and electrical engineering just to name a few. While there are far too many applications to show examples of each, their properties are the same.

A common use for vectors in physics and engineering applications is adding up forces acting on an object.

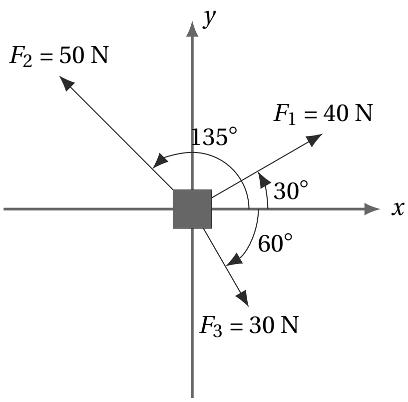


Figure 4.19

Example 4.3.1: – Adding forces

Suppose there are three forces acting on an object as shown in Figure 4.19, a 40 Newton [a] force acting at 30°, a 30 Newton force acting at 300° and a 50 Newton force acting at 135°. Find the resultant force vector acting on the object.

Solution:

The resultant force will be the sum of all the vectors. To add them we will first write them as components. Since we are measuring all the the angles from the horizontal x-axis the signs of each of the components will be correct because the sine and cosine functions will be positive and negative in the correct quadrants. You can verify this by noticing that the x component of F_2 and the y component of F_3 are both negative.

$$\begin{aligned}
\overrightarrow{F_1} &= 40\langle\cos(30°),\sin(30°)\rangle \\
&\approx \langle 34.641, 20\rangle\,\text{N} \\
\overrightarrow{F_2} &= 50\langle\cos(135°),\sin(135°)\rangle \\
&\approx \langle -35.355, 35.355\rangle\,\text{N} \\
\overrightarrow{F_3} &= 30\langle\cos(300°),\sin(300°)\rangle \\
&\approx \langle 15, -25.981\rangle\,\text{N}
\end{aligned}$$

$$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \langle 34.641, 20\rangle + \langle -35.355, 35.355\rangle + \langle 15, -25.981\rangle$$

$$\overrightarrow{R} = \langle 14.286, 29.375\rangle$$

We can find the magnitude

$$\left\|\overrightarrow{R}\right\| = \sqrt{14.286^2 + 29.375^2} \approx 32.664\text{ N}$$

and direction of the resultant vector:

$$\theta = \tan^{-1}\left(\frac{29.375}{14.286}\right) \approx 64°$$

[a] A **Newton** (N) is a metric unit of force $N = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$

Example 4.3.2: – Gravitational Force

A handtruck/dolly is being pulled up a board-ramp inclined 30° with dolly-load combined weight of 250 pounds (operator of the dolly not included). It is known that the board has a maxiumum load carrying capacity of 220 pounds at it's center (weakest point) when horizontal.

a) Will the board/ramp break?

b) What force is required to pull the dolly up the ramp?

Solution:

To begin we first need to draw an illustration of the problem (Figure 4.20). The weight of an object is the gravitational force from which Earth attracts it. The force always acts vertically downward which is indicated by $\|\overrightarrow{w}\|$ in the illustration. The components of $\overrightarrow{w}$ are labelled $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ instead of horizontal,or x, and vertical, or y, components because in this illustration the components aren't horizontal and vertical in standard position; rather we'll refer to them as perpendicular and parallel to the ramp.

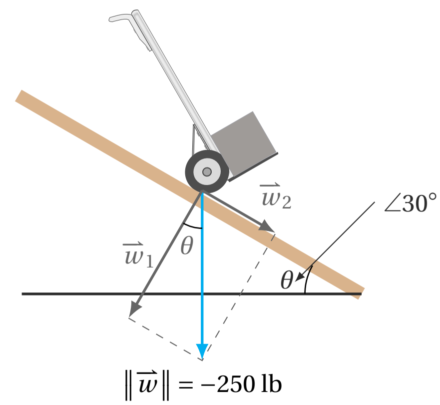


Figure 4.20

a)
If the boards maximum load capacity was determined horizontally then we need to determine the force perpendicular to the ramp which is $\overrightarrow{w}_1$. Notice that $\overrightarrow{w}_1$ is adjacent to θ, thus we have

$$\begin{aligned}\overrightarrow{w}_1 &= \|\overrightarrow{w}\|\cos\theta \\ &= -250\cos 30° \\ &\approx -216.506 \text{ lb}\end{aligned}$$

Since $\overrightarrow{w}$ is vertical then $\|\overrightarrow{w}\|$ is -250 since it's a gravitational force. Also it can be calculated by $250\sin 270°$.

The perpendicular force acting on the ramp is approximately 217 pounds which is under the maximum breaking force, so no the ramp will not break.

b)
The force required to pull the dolly up the ramp has to be greater than the gravitational force pulling it down the ramp. The force down the ramp is parallel to the ramp which is labelled as $\overrightarrow{w}_2$ in Figure 4.20. Notice that $\overrightarrow{w}_2$ is the same length as the opposite side of angle θ.

$$\begin{aligned}\overrightarrow{w}_1 &= \|\overrightarrow{w}\|\sin\theta \\ &= -250\sin 30° \\ &= -125 \text{ lb}\end{aligned}$$

The gravitational force pulling the dolly down the ramp is -125 pounds, thus the force to pull the dolly up the ramp has to be greater (> 125 lb). If the opposite force was equal to the gravitational force the dolly would be stationary.

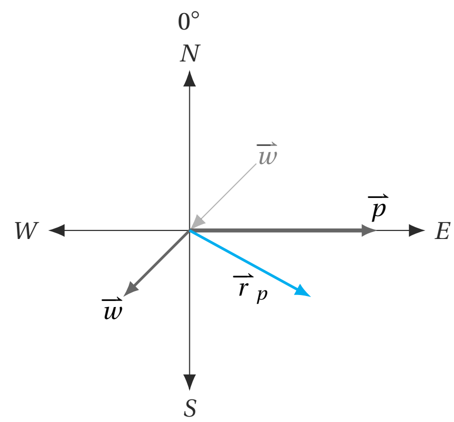


Figure 4.21: Not to scale

Example 4.3.3: – Cardinal Direction

A plane is traveling due East at an airspeed of of 322 km/h. If a wind of 40 km/h is coming in from due Northeast, find the ground speed and bearing of the plane (Figure 4.21.

Solution:

The cardinal direction of NE is 45°, and if a wind is coming in from NE then its heading is due Southwest (SW). To find the resultant magnitude and direction we just need to add the two vectors together. This approach is the same as all others except some may find it easier to setup and add the vector components with a table.

Vector	Horizontal Component	Vertical Component
$\vec{p}$	$322\cos 0° = 322$	$322\sin 0° = 0$
$\vec{w}$	$40\cos 225° = -28.2843$	$40\sin 225° = -28.2843$
$\vec{r}_p$	322+(-28.2843)=293.716	0+(-28.2843)=-28.2843

Now that we have the components of the resultant vector, we can determine the magnitude and direction.

$$\|\vec{r}_p\| = \sqrt{293.716^2 + (-28.2843)^2} \approx 295.074 \text{ km/h}$$

To determine the bearing we first need to know the angle in standard form.

$$\theta_{\vec{r}_p} = \tan^{-1}\left(\frac{-28.2843}{293.716}\right) \approx -5.501°$$

The angle in standard form is approximately $-5.5°$, thus the bearing of the plane is $S\ 95.5°E$.

Example 4.3.4: – Electronics

In a parallel resistance-capacitance (RC) circuit, the current $\vec{I}_C$, or simply I_C, through the capacitance leads the current $\vec{I}_R$, or simply I_R, through the resistance by 90°, as illustrated in Figure 4.22. If $I_C = 0.75$ A and $I_R = 1.5$ A, find the total current $\vec{I}$ in the circuit and the phase angle θ of the circuit.

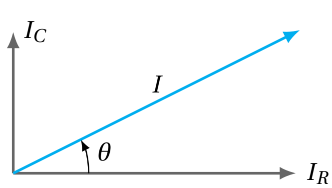


Figure 4.22

Solution:

$$\vec{I} = \sqrt{I_C^2 + I_R^2} = \sqrt{(0.75)^2 + (1.5)^2} \approx 1.68 \text{ A}$$

$$\theta = \tan^{-1}\left(\frac{I_C}{I_R}\right)$$

$$= \tan^{-1}\left(\frac{0.75}{1.5}\right) \approx 26.57°$$

Example 4.3.5: – Frictional forces on inclined plane

A concrete block is at rest on an inclined plane that makes an angle of 30° with respect to the horizontal. If the concrete block weighs 100.0 lb, what is the frictional force between the block and the inclined surface?

Solution:

Here the frictional force F_f is equal but opposite the gravitational force acting on the block to pull it down the plane. The illustration in Figure 4.23 shows F_f equal but opposite the gravitational forces acting on the block. The way the illustration is drawn with labels for two different angles, we have two trigonometric functions for solving the component parallel to the inclined surface.

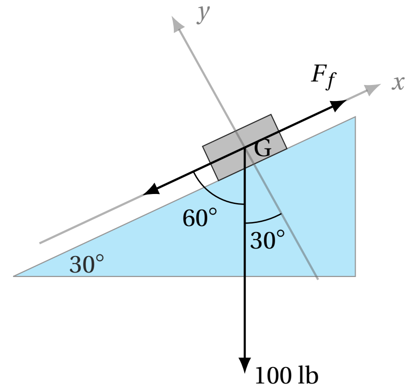


Figure 4.23

$$F_f = 100 * \cos 60^\circ = 50 \text{ lb}$$

or

$$F_f = 100 * \sin 30^\circ = 50 \text{ lb}$$

Example 4.3.6: – Satellite orbit

For a satellite orbiting in a circular motion around the Earth, the tangential component $\vec{a}_T = r\alpha$ and the centriptal component $\vec{a}_R = r\omega^2$ of its acceleration are shown in Figure 4.24. The radius r is the distance from the center of the Earth to the satellite, its angular velocity is ω, and its angular acceleration is α which is the rate that ω is changing.

While in orbit, a satellite travels in a circular path with a radius of 6.789×10^6 m. At this radius $\omega = 1.32 \times 10^{-3}$ rad/s, and $\alpha = (0.450 \times 10^{-6})$ rad/s^2. What is the magnitude of the resultant acceleration and the angle it makes with the tangential component?

Solution:

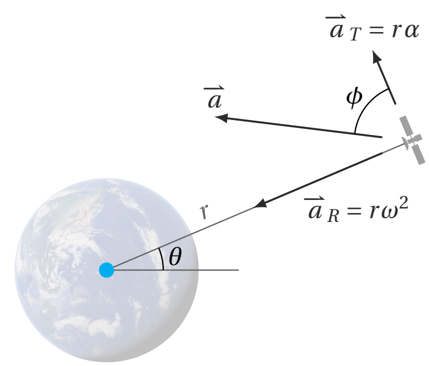


Figure 4.24

$$\begin{aligned}\vec{a}_R &= r\omega^2 = (6.789 \times 10^6)(1.32 \times 10^{-3})^2 \\ &= 11.83 \text{ m/s}^2 \\ \vec{a}_T &= r\alpha = (6.789 \times 10^6)(0.450 \times 10^{-6}) \\ &= 3.05 \text{ m/s}^2\end{aligned}$$

Since a tangent line to a circle is perpendicular to the radius at the point of tengency, $\vec{a}_T$ is perpendicular to $\vec{a}_R$. Thus,

$$\begin{aligned}\vec{a} &= \sqrt{(\vec{a}_T)^2 + (\vec{a}_R)^2} = \sqrt{11.83^2 + 3.05^2} \\ &= 12.22 \text{ m/s}^2 \\ \phi &= \tan^{-1}\left(\frac{\vec{a}_R}{\vec{a}_T}\right) = \tan^{-1}\left(\frac{11.83}{3.05}\right) \\ &= 75.54^\circ\end{aligned}$$

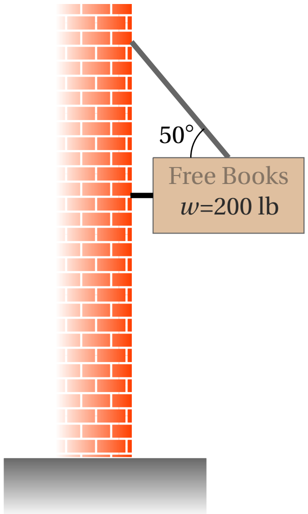


Figure 4.25

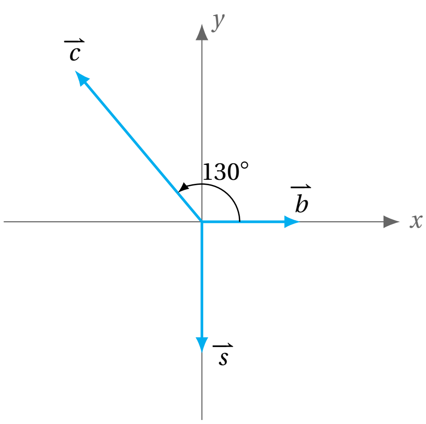


Figure 4.26

Example 4.3.7: – Addition of vectors

A 200 lb sign is held in position by a cable inclined 50° with the horizontal, and a steel brace perpendicular to the wall as shown in Figure 4.25. Find the magnitudes of the forces in the cable and the brace that keep the sign stationary.

Solution:

Drawing a diagram of the vectors in standard position will help greatly with this problem (Figure 4.26). The steel bar holding the sign out from the wall is exerting a horizontal force $\overrightarrow{b}$ in the positive x direction, the cable exertes a pulling force $\overrightarrow{c}$ at 130°, and the weight of the sign is vertically downward with an angle of 270° or −90°, and is represented in the vector illustration as $\overrightarrow{s}$. In this example, we know the directions but not the magnitudes with the exception of the weight of the sign where $\|\overrightarrow{s}\| = -200$ lb.

Vector	Horizontal Component	Vertical Component
$\overrightarrow{c}$	$\|\overrightarrow{c}\|\cos 130^\circ$	$\|\overrightarrow{c}\|\sin 130^\circ$
$\overrightarrow{b}$	$\left\|\overrightarrow{b}\right\|\cos 0^\circ = \left\|\overrightarrow{b}\right\|$	$\left\|\overrightarrow{b}\right\|\sin 0^\circ = 0$
$\overrightarrow{s}$	$\|\overrightarrow{s}\|\cos 270^\circ = 0$	$\|\overrightarrow{s}\|\sin 270^\circ = -200$
Σ	$\overrightarrow{R}_x = \|\overrightarrow{c}\|\cos 130^\circ + \left\|\overrightarrow{b}\right\|$	$\overrightarrow{R}_y = \|\overrightarrow{c}\|\sin 130^\circ - 200$

Since the sign is stationary the components of the resultant vector $\overrightarrow{R}_x$ and $\overrightarrow{R}_y$ can be set equal to zero. Thus we have the following two equations:

$$\overrightarrow{R}_x = \|\overrightarrow{c}\|\cos 130^\circ + \left\|\overrightarrow{b}\right\| = 0 \tag{4.1}$$

$$\overrightarrow{R}_y = \|\overrightarrow{c}\|\sin 130^\circ - 200 = 0 \tag{4.2}$$

From the second equation we obtain the following

$$\begin{aligned}\|\overrightarrow{c}\| &= \frac{200}{\sin 130^\circ}\\ &\approx 261.08 \text{ lb}\end{aligned}$$

Substituting this result into (4.1) we get

$$\begin{aligned}261.08\cos 130^\circ + \left\|\overrightarrow{b}\right\| &= 0\\ \left\|\overrightarrow{b}\right\| &= -261.08\cos 130^\circ\\ &\approx 161.82 \text{ lb}\end{aligned}$$

Therefore, the tension in the cable is 261.08 lb, and the compression force on the bar is 161.82 lb.

4.4 Oblique Triangles: Law of Sines

4.4.1 Law of Sines

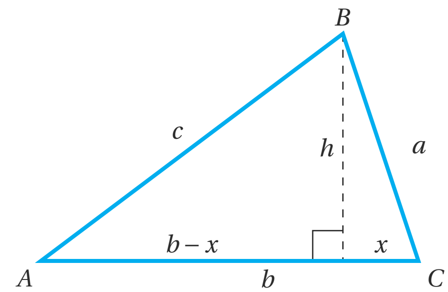


Figure 4.27

Up to now all the triangles we have looked at have been right triangles (one angle of 90°). If we knew two other pieces of information about the triangle, lengths of sides or angle measure, we could solve the triangle. Recall that to solve a triangle we wanted to find the lengths of all the sides and the measure of all the angles. Suppose we have a triangle with no right angles such as $\triangle ABC$ in Figure 4.27. A triangle with no right angles is called an **oblique triangle**. For our oblique triangle we label the angles with upper case letters A, B, and C and the sides opposite those angles with the corresponding lower case letter. Suppose we want to find a relationship between the $\sin A$ and the sides of triangle. We can't use our usual relationship of opposite over hypotenuse because that applies to right triangles. We will draw the height of the triangle h, (in this case from B), and divide the triangle into two right triangles. With the right triangles we can use our usual relationships:

$$\sin A = \frac{h}{c} \qquad \sin C = \frac{h}{a}$$

Solving each of the equations for h gives us

$$h = c\ \sin A \qquad h = a\ \sin C$$

Setting them equal

$$\begin{aligned} h &= h \\ c\ \sin A &= a\ \sin C \\ \frac{\sin A}{a} &= \frac{\sin C}{c} \end{aligned}$$

We can similarly find a relationship for $\sin B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

This is known as the **Law of Sines** and is summarized in the table below.

Law of Sinesθ

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The reciprocal is also true

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Note: The law of sines was proved for an acute triangle where all the angles were less than 90° but the law holds for all triangles.

There are 2 cases where we can use the law of sines. In each of these cases we need three pieces of information.

Case 1: One side and two angles (AAS or ASA)
Case 2: Two sides and an angle opposite one of them (**S**ide **S**ide **A**ngle SSA)

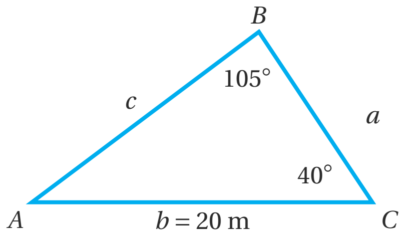


Figure 4.28

Example 4.4.1: – Case 1: One side and two angles (AAS)

Solve the triangle in Figure 4.28 where $B = 105°$, $C = 40°$, and $b = 20$ meters.

Solution:

Recall that to solve the triangle we need to find the remaining sides and angles. We begin with the missing angle because the sum of the angles of a triangle is always 180°.

$$\begin{aligned} A &= 180 - B - C \\ &= 180 - 105° - 40° \\ &= 35° \end{aligned}$$

So $A = 35°$ and by the law of sines we can find the missing sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 35°} = \frac{20}{\sin 105°} = \frac{c}{\sin 40}$$

So we have the following two equations:

$$\frac{a}{\sin 35°} = \frac{20}{\sin 105°} \quad \text{and} \quad \frac{20}{\sin 105°} = \frac{c}{\sin 40}$$

and we can solve for a and c

$$a = \left(\frac{20}{\sin 105°}\right)(\sin 35°) \quad \text{and} \quad c = \left(\frac{20}{\sin 105°}\right)(\sin 40)$$

$$a \approx 11.88 \text{ m} \quad \text{and} \quad c \approx 13.11 \text{ m}$$

4.4.2 The Ambiguous Case: SSA

In Example 4.4.1 we knew two of the angles and one side. This amount of information determines one unique triangle. In the case where you know two sides and an angle opposite one of them there are 3 possible outcomes which are shown in Figure 4.29: no solutions, one solution or two solutions. This is called **the ambiguous case**.

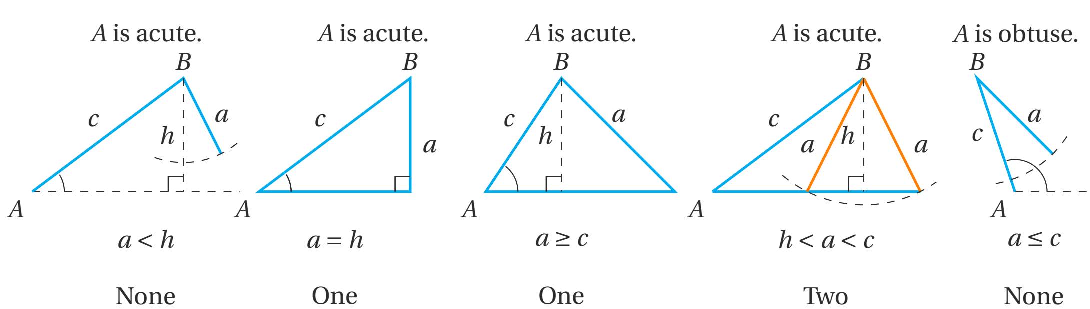


Figure 4.29

Example 4.4.2: – Case 2: Two sides and one angle, two solutions (SSA)

Solve the triangle where $A = 60°$, $a = 9$, and $c = 10$.

Solution:

When you have an angle and two sides you want to draw what you know and then calculate the height. The height will let you know if you can make a triangle or not. The side opposite the angle you know has to be at least as long as the height or you can't make a triangle.

$$\sin 60° = \frac{h}{10} \implies h = 8.66$$

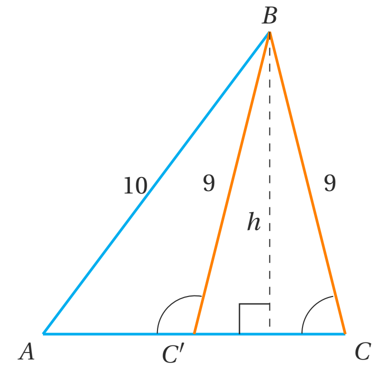


Figure 4.30

In Figure 4.30 the red sides are the two possibilities because

$$(h = 8.66) < (a = 9) < (c = 10)$$

We start by solving the triangle where C is an acute angle. Using the law of sines, $\frac{\sin A}{a} = \frac{\sin C}{c}$, we can solve for C

$$\frac{\sin 60}{9} = \frac{\sin C}{10} \implies C = \sin^{-1}\left(\frac{10 \sin 60}{9}\right) = 74.21°$$

and $B = 180° - 60° - 74.21° = 45.79$. Then the final side can be found with the law of sines again.

$$\frac{9}{\sin 60°} = \frac{b}{\sin(45.79°)} \implies b = \frac{9\sin(45.79°)}{\sin 60°} = 7.45$$

The solution to the first triangle is $\boxed{C = 74.21°,\ B = 45.79° \text{ and } b = 7.45}$.

The second triangle has $C' > 90°$ and is the supplementary to C. (Why?)

$$C' = 180° - 74.21° = 105.79$$

and $B' = 180° - 60° - 105.79° = 14.21°$. The final side can once again be calculated using the law of sines.

$$\frac{9}{\sin 60°} = \frac{b'}{\sin(14.21°)} \implies b' = \frac{9\sin(14.21°)}{\sin 60°} = 2.55$$

The solution to the second triangle is $\boxed{C = 105.79°,\ B = 14.21° \text{ and } b = 2.55}$.

Example 4.4.3: – Case 3: Two sides and one angle, No solution (SSA)

Solve the triangle where $A = 30°$, $a = 6$, and $b = 12.8$.

Solution:

In this case we have no solution because the sides can't meet. Drawing a diagram of the information you know will help to see this as in Figure 4.31. Consider the height h of the this possible triangle.

$$\sin 30 = \frac{h}{12.8} \implies h = 6.4$$

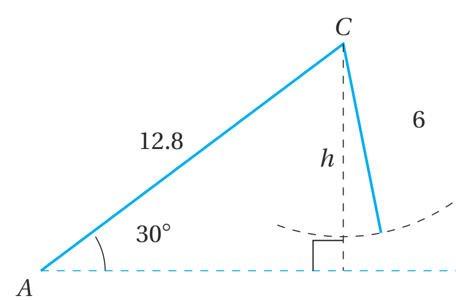


Figure 4.31

Since the height is 6.4 but the side opposite A has length 6, there is no way to construct this triangle and hence there is $\boxed{\text{no solution.}}$

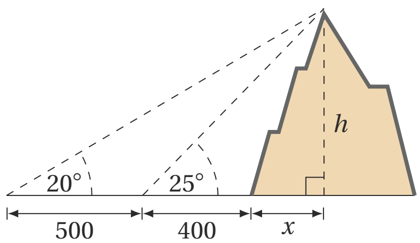


Figure 4.32

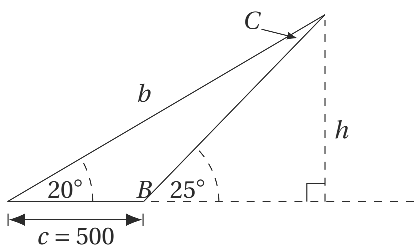


Figure 4.33

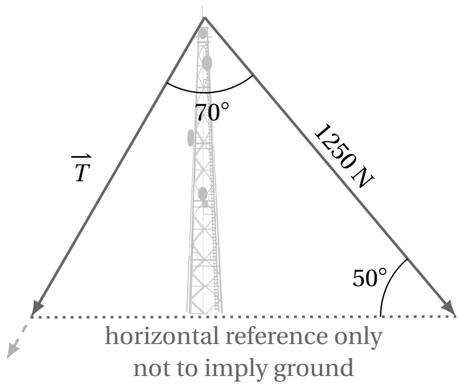


Figure 4.34

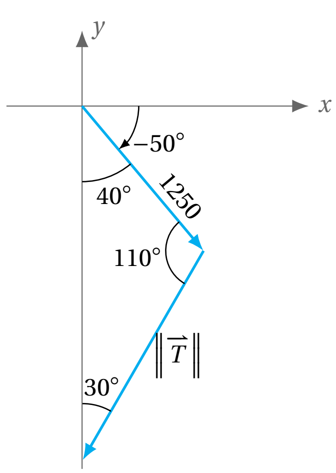


Figure 4.35

Example 4.4.4:

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be 25°. She then walks 500 ft straight back and measures the angle of elevation to now be 20°. How tall is the mountain?

Solution:

We assume that the ground is flat and not inclined relative to the base of the mountain and we let h be the height of the mountain as in Figure 4.32. It will helpful to redraw the diagram (Figure 4.33) without unneeded information, and label the diagram so that the application of the *law of sines* is apparent.

We know that angle B is supplementary to 25° so $B = 180° - 25° = 155°$. The angles in a triangle add up to 180° so $C = 5°$. Now we have enough information to use the law of sines to calculate the distance from the second observation point to the top of the mountain, length b in the diagram.

$$\begin{aligned}\frac{b}{\sin 155°} &= \frac{500}{\sin 5°}\\ b &= \frac{500 \sin 155°}{\sin 5°}\\ b &\approx 2424\text{ft}\end{aligned}$$

Now we can use the right triangle with the height h as the opposite side to the 20° and $b = 2424$ ft as the hypotenuse.

$$h = 2424 \sin 20° = 829\text{ft}$$

This is the same height we had calculated earlier but the calculations were simpler.

Example 4.4.5:

Find the tension $\vec{T}$ in the left guy wire attached to the top of the tower shown in Figure 4.34.

Solution: We must first recognize that we can't just find the lenth of the side in this case because the lengths of the sides aren't given, rather they represent tension. *One* way to solve this is to use the law of sines where we can redraw a diagram with the vectors head-to-tail so that the terminal point of the second vector is vertical to the initial point of the first (Figure 4.35). The reason for this is because the horizontal components must be equal. From here we can use the law of sines to solve for $\left\|\vec{T}\right\|$.

$$\begin{aligned}\frac{\left\|\vec{T}\right\|}{\sin 40°} &= \frac{1250}{\sin 30°}\\ \left\|\vec{T}\right\| &= \frac{1250 \cdot \sin 40°}{\sin 30°}\\ &= 1606.97 \text{ N}\end{aligned}$$

4.5 Law of Cosines

4.5.1 Introduction

In Section 4.4 we were able to solve triangles with no right angles using the law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The law of sines works in two cases:

Case 1: One side and two angles (AAS or ASA)
Case 2: Two sides and an angle opposite one of them (SSA)

There are two cases for which the law of sines does not work because we only have one piece of information in each of our ratios. To use the law of sines you have to have at least one angle and its corresponding opposite side along with one more piece of information being either a side or an angle. However, this is not true for these last two cases.

Case 3: Three sides (SSS)
Case 4: Two sides and the included angle (SAS)

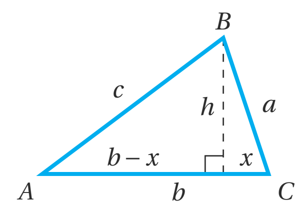


Figure 4.36: Law of Cosines Diagram

To find another equation to solve the last two cases we will once again construct an oblique triangle and label the angles with upper case letters A, B, and C and the sides opposite those angles with the corresponding lower case letter. We draw the height of the triangle h, (in this case from B), and divide the triangle into two right triangles. Now side b is divided into two pieces, one with length x and the other with length $b - x$ (Figure 4.36). Using the Pythagorean theorem we can write an equation for h for both triangles.

For the triangle on the right

$$h^2 = a^2 - x^2 \tag{4.3}$$

For the triangle on the left

$$\begin{aligned} h^2 &= c^2 - (b - x)^2 \\ h^2 &= c^2 - \left(b^2 - 2bx + x^2\right) \\ h^2 &= c^2 - b^2 + 2bx - x^2 \end{aligned} \tag{4.4}$$

Both of these equations involve x but we would like to use only the sides and angles originally given so using the cosine we see that $x = a \cos C$. Now set equation (4.3) equal to equation (4.4) and simplify.

$$\begin{aligned} h^2 &= h^2 \\ a^2 - x^2 &= c^2 - b^2 + 2bx - x^2 \\ c^2 &= a^2 + b^2 - 2bx \end{aligned}$$

Replace $x = a \cos C$

$$c^2 = a^2 + b^2 - 2ab \cos C. \tag{4.5}$$

This is known as the **Law of Cosines** And it relates the three sides of the triangle and one of the angles. This equation can be written in terms of any of the angles. The results are summarized here.

Law of Cosines

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

Standard Form	Alternative Form
$a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\cos B$	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Note: The law of cosines was proved for an acute triangle where all the angles were less than 90° but the law holds for all triangles.

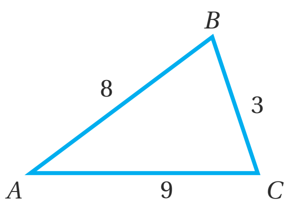


Figure 4.37

Example 4.5.1: – Case 3: Three sides (SSS)

Solve the triangle in Figure 4.37 where $a = 3$, $b = 9$, and $c = 8$.

Solution:

Recall that to solve the triangle we need to find all sides and angles. We have three sides so we can't use the law of sines but we can use the law of cosines. We will use the alternate form so we can find one of the angles. We will start with the largest angle, which is opposite the longest side, $\angle B$.

Note: When using the *law of cosines*, it is helpful to determine the largest angle first. The reason for this is because if cosine is positive, then the angle is acute. If cosine is negative the angle is obtuse. Once the largest angle is determined, then the other two angles must be acute. Also, if cosine is zero then this means the angle is 90°.

$$\begin{aligned}\cos B &= \frac{a^2 + c^2 - b^2}{2ac}\\ &= \frac{8^2 + 3^2 - 9^2}{2\cdot 3\cdot 8}\\ &= -\frac{1}{6}\end{aligned}$$

So, $B = 99.59°$. Generally if you can use the law of sines it is easier than the law of cosines. Now that we have one of our angles we can use the law of sines to find another angle, say $\angle A$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{3} = \frac{\sin 99.59°}{9}$$

$$A = \sin^{-1}\left(\frac{3(\sin 99.59°)}{9}\right)$$

Then $A = 19.19°$ and $C = 180° - A - B = 180 - 19.19° - 99.59° \implies C = 61.22$.

Example 4.5.2: – Case 4: Two sides and the included angle (SAS)

Solve the triangle where $A = 55°$, $b = 3$, and $c = 10$.

Solution: Figure 4.38 is a sketch of the given information. Once again we can't use the law of sines because we don't know an angle and the length of its opposite side. We will start by calculating the length of a with the law of cosines and then use the law of sines to find another angle. While we could use the law of cosines to do solve for the angle, it is easier to use the law of sines whenever you have the choice.

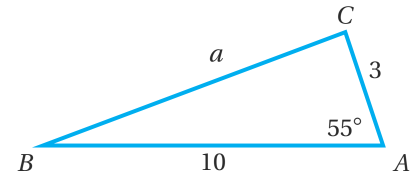


Figure 4.38

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\ A \\ &= 3^3 + 10^2 - 3 \cdot 3 \cdot 10\cos(55°) \\ &= 74.5854 \end{aligned}$$

So $a = 8.64$. Using the law of sines, $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, we can solve for C.

$$\frac{\sin 55°}{8.64} = \frac{\sin C}{10} \implies C = \sin^{-1}\left(\frac{10\sin 55°}{8.64}\right) = 108.48°$$

and $B = 180° - 55° - 108.48° = 16.52°$.

The solution to the first triangle is $C = 108.48°,\ B = 16.52°$ and $a = 8.64$.

Example 4.5.3:

Two radar stations located 10 km apart both detect a UFO located between them. Station Alpha calculates the distance to the object to be 7500 m and Station Beta calculates the distance as 9200 m. Find the angle of elevation measured by both stations (α) and (β). See Figure 4.39

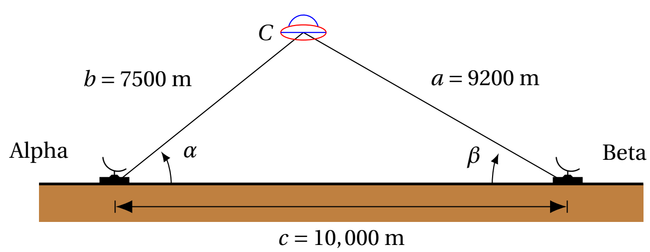


Figure 4.39: UFO and radar stations

Solution:

The triangle formed by the radar stations and the UFO is not a right triangle and we know three sides (SSS). This means we need to use the law of cosines to calculate one of the angles. As before we will use the law of sines to calculate the second angle. Since we are looking for the angle we need the alternate form of the law of cosines:

$$
\begin{aligned}
\cos\beta &= \frac{a^2+c^2-b^2}{2ac} \\
&= \frac{9200^2+10000^2-7500^2}{2(9200)(10000)} \\
&= 0.697772
\end{aligned}
$$

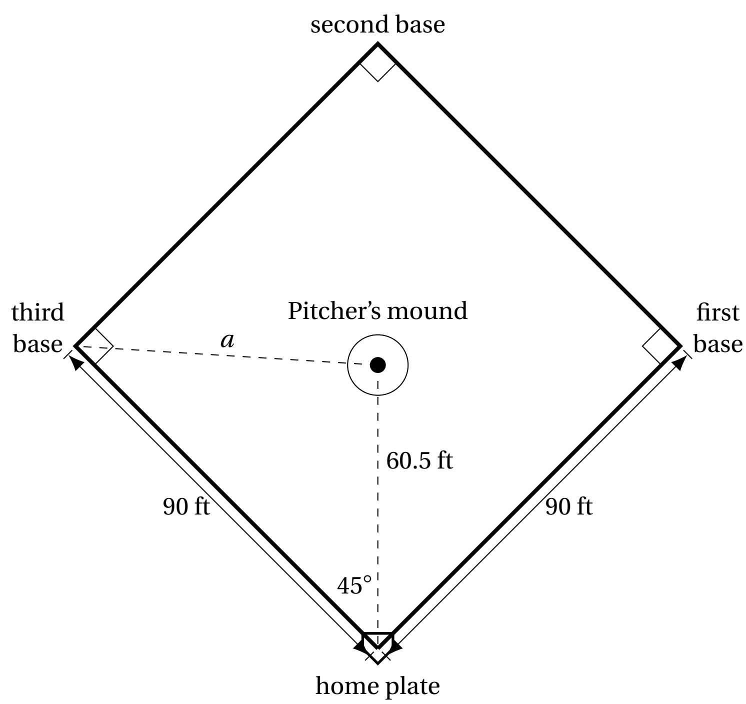
So $\beta = \cos^{-1}(0.697772) = 45.75°$ and we can use the law of sines to find α.

$$
\begin{aligned}
\frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\
\frac{\sin\alpha}{9200} &= \frac{\sin 45.75°}{7500} \\
\sin\alpha &= \frac{9200\sin 45.75°}{7500} \\
\alpha &= 61.48°
\end{aligned}
$$

Then $\alpha = 61.48°$ and $\beta = 45.75°$

Example 4.5.4:

A baseball diamond is a square with 90 foot sides, with a pitcher's mound 60.5 feet from home plate. How far is it from the pitcher's mound to third base? A diagram of the dimensions of a baseball diamond is in the figure below.



Solution:

Solution:

It is tempting to assume the pitcher's mound is in the center of the baseball diamond but it is not. It is located about 3 feet closer to home plate than the center. The distance to third base will therefore be different than the distance to home plate. We do have two sides of a triangle and the angle between them. The triangle is drawn on the diagram and the angle is 45° (why?). Using the law of cosines we can find the missing length.

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
a^2 &= 90^2 + 60.5^2 - 2(90)(60.5)\cos 45^\circ \\
a^2 &= 4060 \\
a &= 63.72 \text{ ft}
\end{aligned}$$

Example 4.5.5: – Displacement

A pengquin and a sloth have escaped from a zoo in Seattle. The penquin takes off in the direction of $N30°E$ for 4 km before being found, while the sloth managed to travel 2 km heading $S40°E$ at the time he was found (Figure 4.40). What was the displacement between the penquin and sloth?

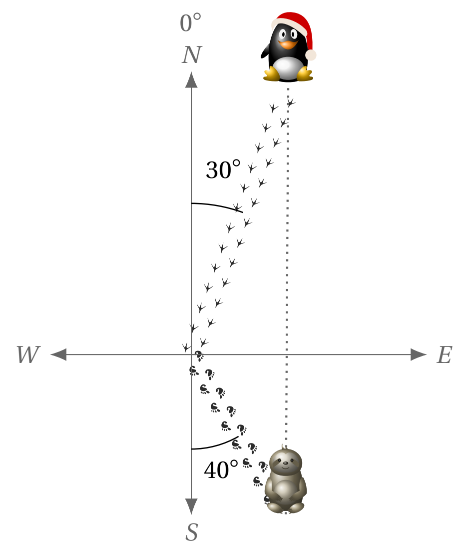


Figure 4.40

Solution:

In the past we've added vectors by components; however, since we are only looking for the displacement, then it will be faster to use the law of cosines in this case since we have a SAS triangle (Figure 4.41 shows the magnitudes of the vectors and the angle between the two paths.

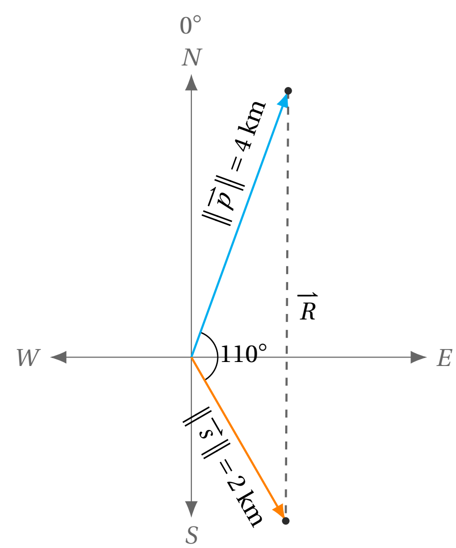


Figure 4.41

$$\begin{aligned}
\left\|\vec{R}\right\|^2 &= \|\vec{p}\|^2 \cdot \|\vec{s}\|^2 - 2\left(\|\vec{p}\|\right)\left(\|\vec{s}\|\right)\cos 110^\circ \\
\left\|\vec{R}\right\|^2 &= 4^2 \cdot 2^2 - 2(4)(2)\cos 110^\circ \\
\left\|\vec{R}\right\| &= \sqrt{4^2 \cdot 2^2 - 2(4)(2)\cos 110^\circ} \\
&= 8.34 \text{ km}
\end{aligned}$$

Notice that the equations for the law of cosines is the same for each side you're solving for. For instance, lets say we're looking for side x knowing its correspoinding opposite angle θ_x and the other two legs of the triangle, say l_1 and l_2. Then the law of cosines is

$$x^2 = l_1^2 + l_2^2 - 2(l_1)(l_2)\cos\theta_x$$

Thus, it's not necessary to remember all 3 equations for a △ABC as if they are somehow different.

Chapter 5

Complex Numbers

5.1 Introduction

Up till now, all numbers that we've interacted with have been *real numbers*. Attempting to describe *imaginary numbers*, or *complex numbers*, is difficult at first because more often than not the first question a student has is "what are the uses, or practical applications for imaginary numbers?" In fact, the term "imaginary" tends to lend itself suspect as if it's reserved for the mathematical purist where its application will undoubtedly resemble fiction. The reality is that imaginary numbers have too many uses and applications to list; however, a generalization of the two main uses seems more appropriate as an introduction.

The first use for imaginary numbers occurs with real quantities that are naturally described with complex numbers. You may have already noticed that both the terms of imaginary, or complex, numbers have seemingly been used interchangeably, and they have been, but more on that later in this section. It is admittedly rare to find many real world applications where complex numbers occur naturally, but they are invaluable in the fields of electronics and electromagnetism. With regard to electronics for instance, the state of a circuit element is described by two real quantities which are the voltage V that runs across it and the current I that flows through it. These two quantities are much more easily described by a single complex number. A circuit element may also have a capacitance C, and an inductance L which essentially is the tendency to resist changes in both voltage and current respectfully that is also represented by a single complex number. With regard to electromagnetism, a single complex number can describe an electromagnetic field that has two real quantities (electric field strength and magnetic field strength) where the electric and magnetic components are real and imaginary components of a complex number respectfully.

The second use for complex numbers, which occurs much more frequently, is with quantities that are defined with real numbers, but are more easily understood with complex numbers. In math and engineering, this occurs all the time in topics of calculus, quadratic equations, differential equations, etc.; however, many reading this chapter may be unfamiliar with examples in those topics, but the use for complex numbers occurs in them nonetheless. When it comes to modeling, say fluid around certain obstacles, complex analysis helps transform a complicated model into a much simpler one. Structural analysis of steel beams in a building, damped oscillator (spring and shocks on a car), statistics and probability theory, and quantum mechanics are all examples of real valued examples made simpler through complex analysis.

5.1.1 Imaginary Unit

The need for complex numbers arose because of certain problems required the need to take the square root of a negative number. We've always been told that we can't take the square root of a negative number; and in fact we been told that the square root of a negative number is undefined. While this is true for real numbers, the square root of a negative number does exist in the complex plane.

Essentially the square root of any number breaks down to the square root of -1. For instance, $\sqrt{-25} = \sqrt{25(-1)} = \sqrt{25}\sqrt{-1} = 5\sqrt{-1}$. The name for the $\sqrt{-1}$ is the **imaginary unit** and is represented as i. Thus, we have

$$i = \sqrt{-1}$$

While the letter i is fairly universal for representing the imaginary unit, some books will opt to let j represent the imaginary unit instead. This is primarily because authors don't like to use the same letter or symbol to represent different things, and i is often used to represent current in electrical systems. In this book, we'll use the letter j to represent the imaginary unit.

As shown above with example of $\sqrt{-25}$, One of the first things we learn to do with working with imaginary numbers is to represent the square root of an imaginary number as the product of a real number and the imaginary unit j. The square root of an imaginary number is called a **pure imaginary number**.

Pure Imaginary Number

If b is a real number greater than zero, $b > 0$, then $\sqrt{-b}$ is a **pure imaginary number** and

$$\sqrt{-b} = \sqrt{b(-1)} = \sqrt{b}\sqrt{-1} = j\sqrt{b}$$

where $j = \sqrt{-1}$

Example 5.1.1:

Express the following square roots in terms of j.

a) $\sqrt{-9}$

b) $\sqrt{-0.25}$

c) $\sqrt{-7}$

Solution:

a) $\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3j$

b) $\sqrt{-0.25} = \sqrt{0.25}\sqrt{-1} = 0.5j$

c) $\sqrt{-7} = \sqrt{7}\sqrt{-1} = j\sqrt{7}$

Note:
In the case of part c) of Example 5.1.1, It is more preferable to write the simplified result as $j\sqrt{7}$ instead of $\sqrt{7}j$. The main reason for doing this is so j is not accidentally interpreted to be under the radical as well such as $\sqrt{7j}$.

5.1.2 Cyclical Nature of Imaginary Numbers

Since $j = \sqrt{-1}$, then there is a pattern when raising j to exponential powers. For instance, notice the pattern below when we raise j to different powers:

$$\begin{aligned}
j &= \sqrt{-1} \\
j^2 &= \left(\sqrt{-1}\right)^2 = -1 \\
j^3 &= \left(\sqrt{-1}\right)^3 = \sqrt{-1}\left(\sqrt{-1}\right)^2 = -j \\
j^4 &= \left(\sqrt{-1}\right)^4 = \left(\sqrt{-1}\right)^2\left(\sqrt{-1}\right)^2 = -1(-1) = 1 \\
j^5 &= \left(\sqrt{-1}\right)^5 = \left(\sqrt{-1}\right)^4\sqrt{-1} = 1 \cdot j = j \\
j^6 &= \left(\sqrt{-1}\right)^6 = \left(\sqrt{-1}\right)^4\left(\sqrt{-1}\right)^2 = 1(-1) = -1 \\
&\vdots
\end{aligned}$$

Because of this cyclical nature of imaginary numbers, any larger powers of j can be reduced to one of the first four basic powers.

Example 5.1.2:

Reduce the following examples to their simplest form:

a) j^{19}

b) j^{1236}

Solution:

Since we know that powers of j are cyclical and repeats after the power of 4, then we need to know how many times that 4 goes into the exponent. Specifically, we want to know the remainder. If the remainder is 1 then our result is the same as $j^1 = j$. If the remainder is 2 then the result is equal to j^2 and so on. If the remainder is zero, then this is the same as j^4 since 4 goes into 4 evenly with a remainder of 0.

a) For j^{19} we have

$$\begin{array}{r} 4 \\ 4\overline{)19} \\ \underline{16} \\ 3 \end{array}$$

Since the remainder is 3, then j^{19} is equivalent to $j^3 = -1$

b) For j^{1236}, we need to do the same thing; however, performing long division is a little more tedious when the powers become large. A calculator is certainly faster when it comes to determining the remainder. To use the calculator, just take 1236 and divide by 4 to get exactly 309. Thus the remainder is 0. Since the remainder is zero, then j^{1236} is equal to $j^4 = 1$.

If the 4 did not divide evenly, and had a decimal value instead. Then to determine the remainder with the calculator, you multiply the decimal part by 4.

In subsection 1.3.2 on page 21. it states that $\sqrt{ab} = \sqrt{a}\sqrt{b}$; however, this simplification of radicals involving an even number of negative values does not apply. Rather, the simplest method to avoid incorrect answers is to let j represent all instances of the $\sqrt{-1}$.

Note: $\left(\sqrt{-a}\right)^2 \neq \sqrt{(-a)(-a)}$

Example 5.1.3:

Simplify $\left(\sqrt{-10}\right)^2$.

Solution:

$$\begin{aligned} \left(\sqrt{-10}\right)^2 &= \left(\sqrt{10}\sqrt{-1}\right)^2 \\ &= \left(j\sqrt{10}\right)^2 \\ &= j^2\left(\sqrt{10}\right)^2 \\ &= (-1)10 \\ &= -10 \end{aligned}$$

Example 5.1.4:

Simplify $\left(2\sqrt{-5}\right)\left(\sqrt{-16}\right)$

Solution:

$$\begin{aligned}
\left(2\sqrt{-5}\right)\left(\sqrt{-16}\right) &= 2\sqrt{-1}\sqrt{5}\cdot\sqrt{-1}\sqrt{16} \\
&= 2j\sqrt{5}\cdot j\sqrt{16} && \text{rewrite } \sqrt{-1} \text{ as } j \\
&= 2j^2\sqrt{5}\cdot 4 \\
&= -8\sqrt{5} && \text{recall } j^2 = -1
\end{aligned}$$

Example 5.1.5:

Simplify $\sqrt{-2}\sqrt{-3}\sqrt{-6}$

Solution:

$$\begin{aligned}
\sqrt{-2}\sqrt{-3}\sqrt{-6} &= \sqrt{-1}\sqrt{2}\cdot\sqrt{-1}\sqrt{3}\cdot\sqrt{-1}\sqrt{6} \\
&= j\sqrt{2}\cdot j\sqrt{3}\cdot j\sqrt{6} && \text{rewrite } \sqrt{-1} \text{ as } j \\
&= j^3\sqrt{2(3)(6)} && \text{Use properties of radicals to rewrite the product of radicals as a single radical.} \\
&= -j\sqrt{36} && \text{recall } j^3 = -j \\
&= -6j
\end{aligned}$$

Below are the components of a complex number:

$$\underset{\text{real}}{a} + \underset{\text{imaginary}}{b}\,j$$

5.1.3 Complex Numbers

A **complex number** occurs when a real number is added to an imaginary number. A complex number is of the form $a + bj$ where both a and b are real numbers, but since b is multiplied by j, we call b the imaginary component. If $a = 0$ then we say that bj is a **pure imaginary number**. When $b = 0$ then we get a number of the form a, which is a real number.

Rectangular Form of a Complex Number

The form $a + bj$ is called the **rectangular form** of a complex number, where a is the **real component** and b is the **imaginary component**.

Two complex numbers are equal if both their corresponding components are equal. In other words, two complex numbers are equal if their real components are equal and their imaginary components are equal.

Equality of a Complex Number

If $a + bj$ and $c + dj$ are two complex numbers, then $a + bj = c + dj$ if and only if $a = c$ and $b = d$.

Example 5.1.6:

$a+bj=2+3j$ if $a=2$ and $b=3$

Example 5.1.7:

Solve the following for both x and y:

$$4x+3j \;=\; 3-x-yj$$

Solution:

To begin we need to identify the parts that are real and imaginary. The easiest way to see what is imaginary or not, is to look for the imaginary numbers (any number multiplied by j). Thus, there are only two imaginary numbers which are $3j$ and $-yj$. It also may help grouping the real components and imaginary components within parenthesis.

$$4x+3j=3-x-yj$$
$$4x+3j=(3-x)-yj$$

parenthesis aren't necessary here, but they do help to keep things ordered

The definition states that two complex numbers are equal if their real components and imaginary components are equal. Thus we have two equations to solve once we set up the real components and the imaginary components equal to each other.

real-components:

$$\begin{aligned} 4x &= (3-x) \\ 4x+x &= 3 \\ 5x &= 3 \\ x &= \frac{5}{3} \end{aligned}$$

imaginary-components:

$$\begin{aligned} -y &= 3 \\ y &= -3 \end{aligned}$$

notice that we don't need to include j.

5.1.4 Conjugate of a Complex Number

Every complex number has a complex conjugate. The **complex conjugate** of a $a+bj$ is $a-bj$. The imaginary component is the only number that changes sign when defining the complex conjugate.

Example 5.1.8:

a) The conjugate of $3+2j$ is $3-2j$

b) The conjugate of $-3-2j$ is $-3+2j$

c) The conjugate of $5j$ is $-5j$

d) The conjugate of 7 is 7

For a quadratic equation of the form $a^2 + bx + c = 0$, the formula for solving the x-intercepts (roots) is given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5.1.9:

Use the quadratic formula to solve $x^2 + 2x + 4 = 0$

Solution:

Substitute the coefficients of $x^2 + 2x + 4$ into the quadratic formula which is shown in the margin.

$$x^2 + 2x + 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

substitute the coefficients into the quadratic formula.

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

simplify some, but don't attempt to simplify too much in one step.

$$= \frac{-2}{2} \pm \frac{\sqrt{(-1)(4)(3)}}{2}$$

separate the fraction into two, and rewrite -12 as a product of factors using perfect squares when possible.

$$= -1 \pm \frac{\sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{3}}{2}$$

$$= -1 \pm \frac{\cancel{2} j\sqrt{3}}{\cancel{2}}$$

simplify further and cancel any like terms

$$= -1 \pm j\sqrt{3}$$

Thus our two roots are $-1 + j\sqrt{3}$, and $-1 - j\sqrt{3}$.

The solutions to an equation equal to zero are often called **zeros**, or **roots**. Notice that the solutions to the equation are $-1 + j\sqrt{3}$ and $-1 - j\sqrt{3}$ in example 5.1.9 above are conjugates of each other. This is not a coincidence, rather complex roots always appear in conjugate pairs. This fact is extremely helpful in applications involving complex numbers. In addition, the product of two conjugate pairs always results in a real number; which is particularly useful when manipulating complex expressions as we'll see in the next section.

Though we probably feel like we have a pretty good grasp of what a complex number is at this point, I'd have to say that the concept of imaginary numbers, imaginary added to real numbers to create what is called complex numbers probably still feels like a mathematical abstraction. We do know and understand the *cyclical* nature of j raised to progressively higher powers which is important. Also we understand how to algebraically manipulate certain complex expressions. In the next section we introduce operations of complex numbers such as adding, subtracting, multiplying, and dividing and their meaning.

5.2 Operations with Complex Numbers

5.2.1 j - Geometrically

Before we begin operations on complex numbers, it's important to have a better understanding of the imaginary unit j.

From the previous section we understand that $j^2 = 1$, or $j = \sqrt{-1}$, and have termed this number *imaginary* as if it doesn't exist. The reality is that imaginary numbers are just as normal as any other number we've worked with in the past such as whole numbers, integers, rationals, and negative numbers. The biggest difference is that when we look at an equation such as

$$x^2 = 1$$

We can solve this easily by taking the square root of both sides of the equation to get the result of both 1 and -1. However, if we introduce the following equation

$$x^2 = -1$$

We stop for a minute and realize there there are no two numbers when multiplied together to result in -1 which is true. This is because we were looking for legitimate solutions involving real numbers only. To better understand what's going on with these two equations, lets break them down and look at them again.

When we solve the equation $x^2 = 1$, what we really have is the following expression

$$1 \cdot x \cdot x = 1$$

and are asking what transformation applied to x twice results in 1 which we know is $x = -1$ and $x = 1$. However, if we look at $x^2 = -1$ in the same way we have

$$1 \cdot x \cdot x = -1$$

and ask the same thing. What transformation applied to x twice results in -1? After examining we conclude that $x = 1$ won't work because the result is a positive 1, and $x = -1$ doesn't work because $1 \cdot -1 = -1$, but multiply that with another -1 flips the result back to a positive 1. The issue here is that we continue to find the result on the real number line where we "flip" back and forth between positive and negative real numbers. What we haven't considered is that instead of flipping, what about a rotation? If we think of x being a rotation then instead of flipping to the opposite side of 0, rather it is rotated about 0.

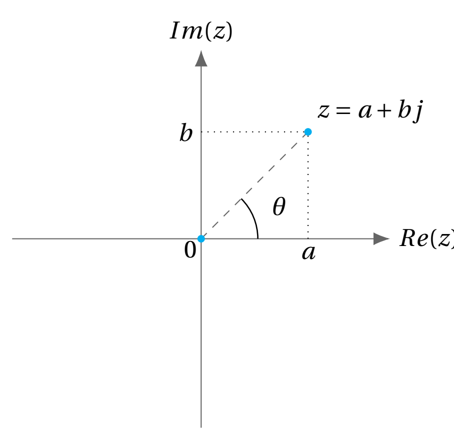


Figure 5.1: Complex Plane

This introduces the concept of the **complex plane** since you can't rotate something about 0 with it not passing through intemediate values. In the complex plane, the horizontal axis is the real axis which is the same number line we're used to. The vertical axis is the imaginary axis as shown in Figure 5.1 along with a random complex number plotted for reference.

Now, if we imagine x being a rotation of 90° about the origin and applying it twice, then we get a 180° rotation. This is illustrated in Figure 5.2. In addition, if we multiply 1 by $-j$, then we get a clockwise (negative) rotation.

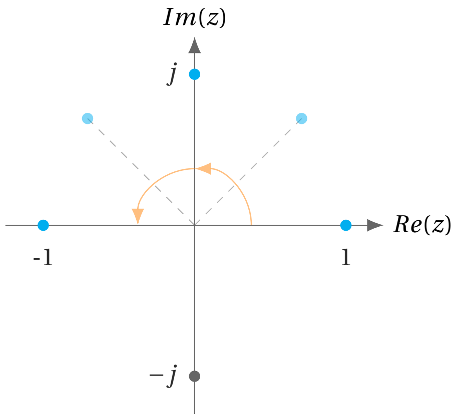


Figure 5.2

Again, If we multiply 1 by j we get a positive rotation (counter clockwise) by 90° , and by multiplying 1 by $-j$ we have a negative rotation (clockwise) of $-90°$. Thus, by applying this multiplication twice, recall $x^2 = x \cdot x$, then we can have a positive rotation of 180° to get -1 since the first multiplication of j gives us a 90° rotation to j on the imaginary axis, and by multiplying again by j gives -1 on the real axis. The same holds true in the negative rotation as well, thus we have two ways to get -1. Therefore, the two square roots of -1 is j and $-j$ or we can write it as $\pm j$.

$$
\begin{aligned}
x^2 &= -1 \\
x &= \pm\sqrt{-1} \\
&= \pm j
\end{aligned}
$$

Recall the cyclical pattern of $j = \sqrt{-1}$ (below for convenience) that the pattern rotates from j to -1, then $-i$, and finally to 1 before repeating. After thinking of, say j^3 as $j \cdot j \cdot j$, it's now understandable geometridally that with every multiple of j, j is rotated by 90° in the complex plane.

$$
\begin{aligned}
j &= \sqrt{-1} \\
j^2 &= \left(\sqrt{-1}\right)^2 = \boxed{-1} \\
j^3 &= \left(\sqrt{-1}\right)^3 = \sqrt{-1}\left(\sqrt{-1}\right)^2 = \boxed{-j} \\
j^4 &= \left(\sqrt{-1}\right)^4 = \left(\sqrt{-1}\right)^2\left(\sqrt{-1}\right)^2 = -1(-1) = \boxed{1} \\
j^5 &= \left(\sqrt{-1}\right)^5 = \left(\sqrt{-1}\right)^4\sqrt{-1} = 1 \cdot j = \boxed{j} \\
j^6 &= \left(\sqrt{-1}\right)^6 = \left(\sqrt{-1}\right)^4\left(\sqrt{-1}\right)^2 = 1(-1) = \boxed{-1} \\
&\vdots
\end{aligned}
$$

Recall that **complex numbers** are numbers that are both real and imaginary. These numbers can occur anywhere in the complex plane in the form of $a + bj$ where a is real and b is imaginary. For example, Figure 5.3 shows the complex number $z = 1 + j$ plotted on the complex plane.

So how does all this help us understand how to add, subtract, multiply, and divide complex numbers? In short, it doesn't yet, but it will help us understand what occurs when we perform those operations on complex numbers. Recall that complex numbers can be used to simplify otherwise complicated tasks that occur with real application. To affectively use complex numbers, we must have a basic understanding of what happens when two complex numbers are combined by adding, multiplying, etc. The algebraic process of combining complex numbers is fairly straight forward.

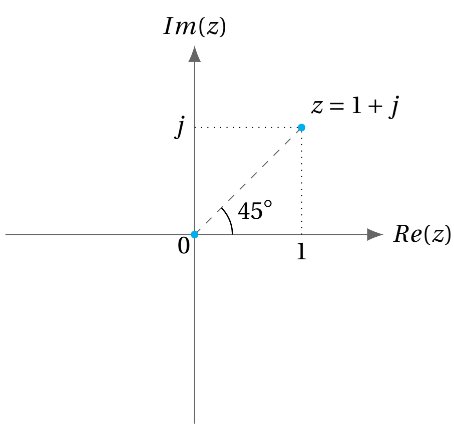


Figure 5.3

5.2.2 Adding, Subtracting Complex Numbers

To add two complex numbers we just need to add the real components together, and add the imaginary components together.

The trick to all complex arithmetic (add, subtract, multiply, divide) is that you can treat j as a variable, but everytime you see j^2 replace it with -1.

Addition of Complex Numbers

If $a+bj$ and $c+dj$ are two complex numbers, then their sum is defined as

$$(a+bj) + (c+dj) = (a+c)+(b+d)j$$

Example 5.2.1: –Adding complex numbers

Find each of the following sums:

a)

$$\begin{aligned}(2+j)+(3+4j) &= (2+3)+(1+4)j \\ &= 5+5j\end{aligned}$$

b)

$$\begin{aligned}(-3+7j)+(-2+4j) &= (-3+(-2))+(7+4)j \\ &= -5+11j\end{aligned}$$

c)

$$\begin{aligned}(-2-3j)+(-6+\sqrt{-4}) &= (-2-6)+(-3+2)j \qquad \sqrt{-4}=\sqrt{4}\sqrt{-1}=2j \\ &= -8-j\end{aligned}$$

It appears that adding complex numbers is straight forward enough, but geometrically what occurs when one complex number is added to another? For instance, graphically how does adding $-2+j$ affect $3+4j$ when added together?

First, we can see that $(3+4j)+(-2+j) = 1+5j$, but lets take a look at the solution in the complex plane shown in figure 5.4.

It appears that when adding $(3+4j)+(-2+j) = 1+5j$ that from $3+4j$ the complex number has shifted exactly -2 units on the real axis and 1 unit up on the imaginary axis. Also, its not difficult to see how adding complex numbers resembles adding vectors. For this reason, complex numbers are treated as vectors with some exceptions.

Vectors have direction and magnitude only, thus can be moved anywhere in the plane without affecting those two attributes. On the other hand, complex numbers always originate from the origin. There are similarities between the two which shouldn't be a surprise. Vectors contain two component x and y, where complex contain two components as well; however, there are limitations.

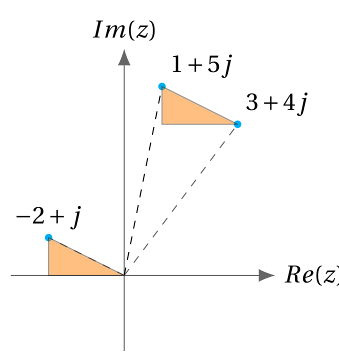


Figure 5.4

To subtract two vectors, we perform the same operation as addition except the real components are subtracted, and the imaginary components are subtracted. The only part that we have to watch out for is when we subtract a negative which becomes positive.

Addition of Complex Numbers

If $a+bj$ and $c+dj$ are two complex numbers, then their difference is defined as

$$(a+bj)\ -\ (c+dj)\ =\ (a-c)+(b-d)j$$

Example 5.2.2:

Find each of the following differences:

a)

$$\begin{aligned}(7+2j)-(3+4j) &= (7+2j)+(-3-4j) \\ &= (7-3)+(2-4)j \\ &= 9-2j\end{aligned}$$

It may be helpful to distribute the negative first.

b)

$$\begin{aligned}(-2-3j)-(-4-5j) &= (-2-3j)+(4+5j) \\ &= (-2+4)+(-3+5)j \\ &= 2+9j\end{aligned}$$

c)

$$\begin{aligned}(a-bj)-(-c+dj) &= (a-bj)+(c-d)j \\ &= (a+c)+(-b-d)j\end{aligned}$$

It is often helpful to rewrite the difference of two complex numbers as the addition of two complex numbers by first distributing the negative throughout the complex number. This will help avoid confusion because it is easy to forget that you are subtracting both components.

5.2.3 Multiplication of Complex Numbers

Multiplying complex numbers is done in the same manner that we multiplied to polynomials together (FOIL method). We take each component in the first complex number and multiply it by both components in the second complex number, then simplify by combining real components together and imaginary components together. Recall that we call this *distributing*. (see Section 1.7)

Multiplication of Complex Numbers

If $a+bj$ and $c+dj$ are two complex numbers, then their product is defined as

$$(a+bj)\dot{(}c+dj) = (ac-bd)+(ad+bc)j$$

While the definition is nice to have, memorizing it isn't recommended. Rather is much easier to simply multiply everything out step-by-step and simplify as shown in the next example.

Example 5.2.3:

Multiply and simplify the answer in the form $a+bi$.

$$(3+4j)(-2+j)$$

Solution:

$$\begin{aligned}(3+4j)(-2+j) &= 3(-2)+3j+4j(-2)+4j^2 \\ &= -6+3j-8j-4 && \text{recall } j^2=-1 \\ &= (-6-4)+(3-8)j \\ &= -10-5j\end{aligned}$$

Example 5.2.4:

Multiply and simplify the answer in the form $a+bj$.

$$(2+3j)(1+j)$$

Solution:

$$\begin{aligned}(2+3j)(1+j) &= 2(1)+2j+3j(1)+3j^2 \\ &= 2+5j-3 \\ &= -1+5j\end{aligned}$$

To see whats going on geometrically when two complex number are multiplied, we'll look at the last example, Ex. 5.2.4. Figure 5.5 shows both complex numbers plotted in the complex plane along with their product. At first glance there doesn't appear to be much of a relationship between the independent complex numbers and their product; however, looking at Figure 5.6 we can see that $(2+3j)$ has been rotated about the origin by the same degree measure that $(1+j)$ is which is 45°.

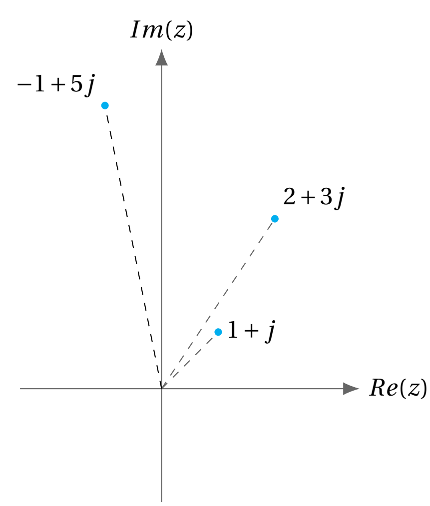


Figure 5.5

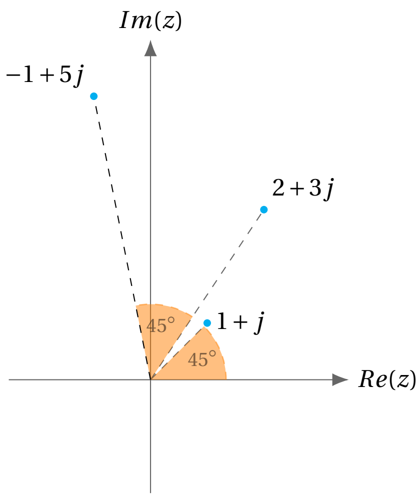


Figure 5.6

The length, or **modulus** is defined as the absolute value of $a+bj$. In other words, it is the distance from the origin.

Length of a complex number

If $a+bj$ is any complex number, then the absolute value, also called the modulus, is given by

$$\left|a+bj\right| = \sqrt{a^2+b^2}$$

Example 5.2.5:

Show that the magnitude of the product of two the complex numbers in Ex. 5.2.4 is equivalent to the product of the individual magnitudes.

Solution:

$$\begin{aligned} \left|1+j\right| &= \sqrt{1+1} = \sqrt{2} \\ \left|2+3j\right| &= \sqrt{2^2+3^2} = \sqrt{13} \\ \left|(1+j)(2+3j)\right| &= \left|-1+5j\right| = \sqrt{(-1)^2+5^2} = \sqrt{26} \end{aligned}$$

$$\sqrt{2}\sqrt{13} = \sqrt{26}$$

Now, what occurs when a complex number is multiplied by its conjugate. Earlier we stated that the product of any complex number and its conjugate results in a real number.

Example 5.2.6:

Find the product of $a+bj$ and its conjugate $a-bj$

Solution:

$$\begin{aligned} (a+bj)(a-bj) &= a^2 - abj + abj - b^2 j^2 \\ &= a^2 - (-1)b^2 \\ &= a^2 + b^2 \end{aligned}$$

Since a, and b are real numbers then the sum of a^2+b^2 is also real.

Example 5.2.7:

Find the product of $-3+4j$ and its conjugate $-3-4j$.

Solution:

$$\begin{aligned}(-3+4j)(-3-4j) &= (-3)^2 + 12j - 12j - 4(4)j)^2 \\ &= 9 - 16(-1) \\ &= 9 + 16 \\ &= 25\end{aligned}$$

5.2.4 Division of Complex Numbers

At first glance, division of complex numbers appears a little complicated. The best approach is to not use the formula given below, rather work out the division process by steps. The main reason the formula appears complicated is that the variables a, b, c, and d aren't able to be combined and simplified throughout the division process whereas a specific example can be, as we'll see in the first example.

We have no way to divide by j so we need to make the denominator a real number. To do that we need to multiply the denominator by its complex conjugate.

Division of Complex Numbers

If $a+bj$ and $c+dj$ are any two complex numbers, then their quotient is given by

$$\frac{a+bj}{c+dj} = \frac{ac+bd}{c^2+d^2} + \frac{bd-ad}{c^2+d^2}\, j$$

Example 5.2.8:

Divide and express the result in the form of $a+bj$.

$$\frac{2+3j}{1+j}$$

Solution:

$$\frac{2+3j}{1+j} = \frac{2+3j}{1+j} \cdot \frac{1-j}{1-j}$$

multiply the numerator and denominator by the conjugate of the denominator. This will replace the complex number in the denominator with a real number.

$$= \frac{2(1) - 2j + 3j - 3j^2}{1^2 - j + j - j^2}$$

FOIL both the numerator and denominator

$$= \frac{(2+3) + (-2+3)j}{1+1}$$

simplify and group real and imaginary components

$$= \frac{5+j}{2}$$

combine like terms

$$= \frac{5}{2} + \frac{1}{2}j$$

rewrite in the from of real plus complex, $a+bj$.

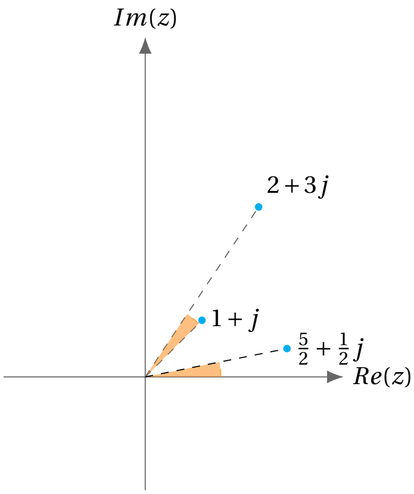


Figure 5.7

Graphically we saw that the product of two complex numbers rotated the one of the complex numbers by the equivalent of the rotation of the other, and the modulus of the result of that product was equivalent to the product of the individual modulus'. Figure 5.7 illustrates what occurs when two complex numbers are divided.

Studying Figure 5.7 we can see that the resultant angle is equivalent to subtracting the angle of $(1+j)$ which is $45°$ from the angle of $(2+3j)$. Moreover, by taking the difference between these two angles, this gives the angle between $(2+3j)$ and $(1+j)$.

The modulus of the result of the quotient is the same as quotient of the individual modulus'. Thus, we have

$$\left|\frac{a+bj}{c+dj}\right| = \frac{|a+bj|}{|c+dj|}$$

Example 5.2.9:

Divide and express the result in the form of real + complex.

$$\frac{a+bj}{c+dj}$$

Solution:

$$\frac{a+bj}{c+dj} = \frac{a+bj}{c+dj}\cdot\frac{c-dj}{c-dj}$$ multiply the numerator and denomintator by the conjugate of the denominator.

$$= \frac{ac-adj+bcj-bdj^2}{c^2-cdj+cdj-d^2j^2}$$ foil numerator and denominator

$$= \frac{(ac+bd)+(bc-ad)j}{c^2+d^2}$$ simplify and group real, and complex values

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}j$$ rewrite in the form of real + complex.

Example 5.2.10:

In an ac circuit, the formula $V = ZI$ relates the voltage V, to impedance Z and current I. Use the formula to find the impedance in a circuit where the voltage is given by $115+11j$, and current is $9+3j$.

Solution:

To begin, we first notice that we must solve the formula given for impedance Z which is straight forward since we only need to divide both sides of the equation by current I.

$$Z = \frac{V}{I}$$

Now, we substitute the values that we have for both voltage and current and simplify.

$$\begin{aligned}
\frac{115+11j}{9+3j} &= \frac{115+11j}{9+3j}\cdot\frac{9-3j}{9-3j}\\
&= \frac{115(9)-115(3)j+11(9)j+11(-3)j^2}{9^2+3^2}\\
&= \frac{(1035+33)+(-345+99)j}{90}\\
&= \frac{1068}{90}+\frac{-246}{90}j\\
&\approx 11.87-2.73j
\end{aligned}$$

Most scientific calculators have the ability to perform operations on complex numbers. In most cases the numbers are entered in the same way their presented on paper. However, since a complex number is defined by having both a real component and an imaginary component, then parenthesis are needed to denoted each number. This is especially true when subtracting, multiplying, and dividing. Figure 5.8 shows the input and output of Ex. 5.2.10 using a scientific calculator.

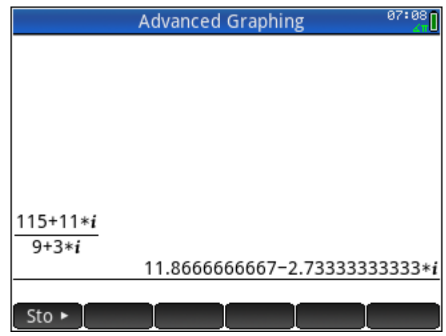


Figure 5.8

5.3 Polar Form of Complex Numbers

5.3.1 Plotting a Complex Number

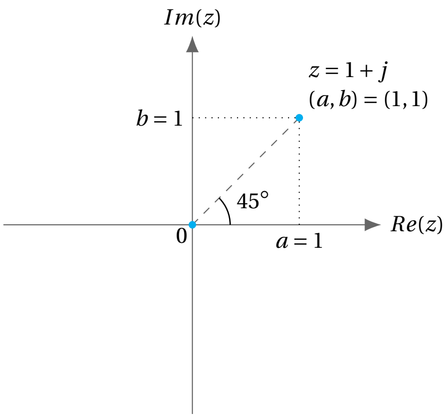


Figure 5.9

Up till this point all complex numbers plotted in the complex plane have been done in the form that's called **rectangular form**.The rectangular form of plotting a complex number is, as you may have guessed from the previous section, performed just as any point would be plotted in the Cartesian plane where x is the real axis, and y is the imaginary axis. In Figure 5.9 where $z = 1 + j$ is plotted you'll notice that along the vertical axis that j is not shown. The reason for this is that j is implied, and that the only values that would be placed along the vertical axis are the coefficients of j. In addition, a line from the origin is not typically drawn unless the complex number is representing a vector. In Figure 5.9, the dashed line is placed to show the angle of rotation from the positive real axis. Figure 5.10 shows several complex points plotted without all the references to direction and such.

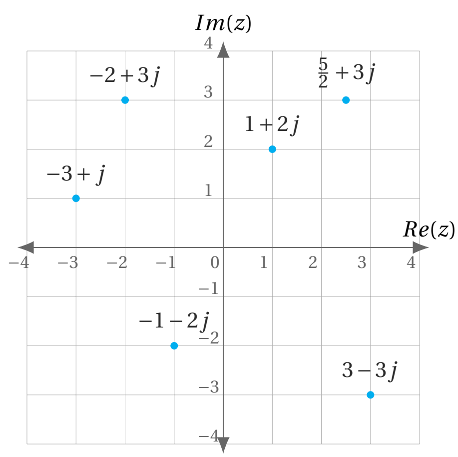


Figure 5.10

5.3.2 Polar Form of a Complex Number

The rectangular form of plotting complex coordinates is not the only method to plotting points in a plane. In Figure 5.11 the angle that $a + bi$ makes with the positive real axis is called the **argument**. As stated in the previous section, the absolute value of a complex number is the distance it is from the origin and is called the **modulus**. In instances where a complex number is treated as a vector it's still referred to as the magnitude. In either case, whether we refer to the length as the magnitude, absolute value, norm, or modulus, the equation to determine it is the same.

Absolute Value of a Complex Numbers

If $a+bj$ is any complex number, then the distance it is from the origin is called the **absolute value of a complex number**, and is denoted $|a+bj|$ with the value

$$|a+bj| = \sqrt{a^2+b^2}$$

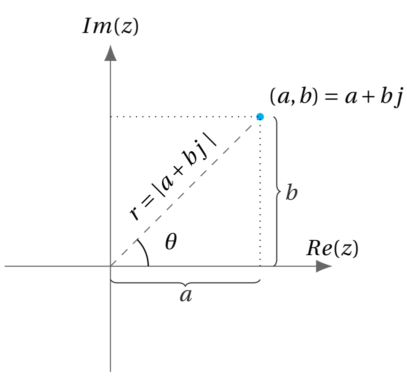


Figure 5.11

After studying Figure 5.11 we see that any complex number exhibits the same properties that a real point does being plotted in the Cartesian plane which is the *rectangular* form from which a triangle can be visualized or extracted. For this reason, a few familiar relationships enable us to make use of trigonometric functions. From our definitions of trigonometric functions, we see that $\cos\theta = \dfrac{a}{r}$, and $\sin\theta = \dfrac{b}{r}$. Solving each of these for a and b respectively gives

$$a = r\cos\theta \tag{5.1}$$

$$b = r\sin\theta \tag{5.2}$$

In addition to the trigonometric expressions for the sides of the triangle, we know that there are several ways to determine the angle θ; however, typically just recalling tangent is sufficient. Also, the length of the radius r is the same as the absolute value of the complex number.

$$\tan\theta = \frac{b}{a} \tag{5.3}$$

$$r = \sqrt{a^2 + b^2} \tag{5.4}$$

Together, the first two equations, 5.1 and 5.2, substituted in for a and b in $a + bj$ and simplifying forms the following

$$a + bj = r\cos\theta + ir\sin\theta = r(\cos\theta + j\sin\theta)$$

The expression $r(\cos\theta + j\sin\theta)$ is called the **polar form of a complex number**, and sometimes referred to as the **trigonometric form of a complex number**. The expression $r(\cos\theta + j\sin\theta$ is often abreviated as r cis θ, or $r\underline{/\theta}$. For r cis θ, c is the abbreaviation for cosine, i is the imaginary unit, and s is the abbreviation for sine. The abbreviation $r\underline{/\theta}$ is read "r at angle θ."

Converting Complex numbers from Polar to Rectangular Form

A complex number written in one of the following polar forms

$$r(\cos\theta + i\sin\theta) \quad r\underline{/\theta} \quad r \text{ cis } \theta$$

has the rectangular coordinates $a + bj$, where

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

Example 5.3.1:

Convert $3(\cos 76° + j\sin 76°)$ to rectangular form, and locate it in the complex plane.

Solution:

$$\begin{aligned} a &= r\cos\theta \\ &= 3\cos 76° \\ &= 0.7258 \end{aligned}$$

$$\begin{aligned} b &= r\sin\theta \\ &= 3\sin 76° \\ &= 2.9109 \end{aligned}$$

Thus, in rectangular form we have $a + bj = 0.7258 + 2.9109j$. Plotting complex numbers are often easier by converting to rectangular form first. $0.7258 + 2.9109j$ is shown in Figure 5.12.

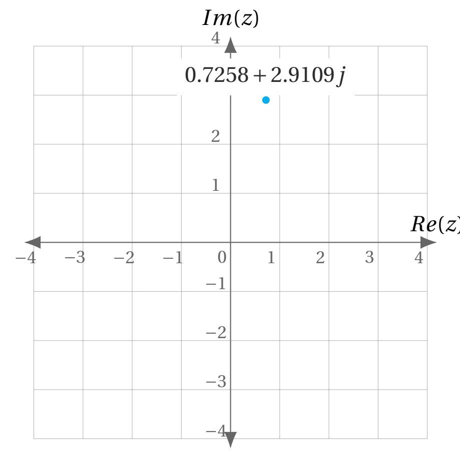


Figure 5.12

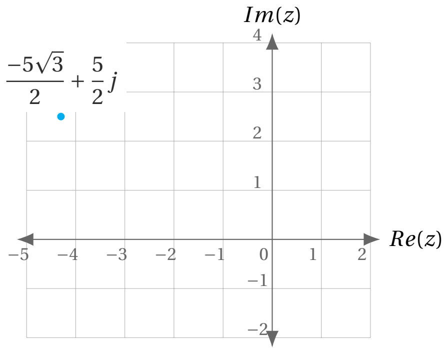


Figure 5.13

Example 5.3.2:

Convert $5\underline{/150^\circ}$ to rectangular form, and locate it in the complex plane.

Solution:

$$\begin{aligned} a &= r\cos\theta \\ &= 5\cos 150^\circ \\ &= \frac{-5\sqrt{3}}{2} \\ &\approx -4.33 \end{aligned}$$

$$\begin{aligned} b &= r\sin\theta \\ &= 5\sin 150^\circ \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

Thus, $a+bi = \frac{-5\sqrt{3}}{2}+\frac{5}{2}j$ and is also plotted in the complex plane in Figure 5.15.

Converting Complex numbers from Rectangular Form to Polar Form

A complex number written in rectangular form as

$$a+bj$$

has the polar form coordinates in the following forms

$$r(\cos\theta + j\sin\theta) \quad r\underline{/\theta} \quad r\text{ cis }\theta$$

where

$$r=\sqrt{a^2+b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Example 5.3.3:

Convert $3-4j$ to polar form and represent it in each of the three forms.

Solution:

First, begin by finding r, then determine the angle θ.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{3^2+(-4)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{-4}{3} \\ &\approx -53.13^\circ \end{aligned}$$

Thus $3-4j$ is equivalent in polar form to $5(\cos -53.13^\circ + j\sin -53.13^\circ)$, or $5 \text{ cis } -53.13^\circ$, or $5\underline{/-53.13^\circ}$. Note, the positive coterminal angle of 306.87° could also have been used.

Example 5.3.4:

The complex number $V = 24.7-66.3j\ V$ represents the voltage in an ac circuit. Express this in polar form.

Solution:

As in the previous example, begin by finding r, then determine the angle θ.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{24.7^2+(-66.3)^2} \\ &= \sqrt{70.75} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{-66.3}{24.7} \\ &\approx -69.6^\circ \end{aligned}$$

Since V is in the fourth quadrant, then representing $\theta = -69.6^\circ$ is okay as it is; however, if we'd prefer to represent θ using positive angles, then the positive coterminal angle is $-69.6^\circ + 360^\circ = 290.4^\circ$. V is shown in figure 5.14.

Thus in polar form we have $V = 70.75\underline{/-69.6^\circ} = 70.75\underline{/290.4^\circ}$.

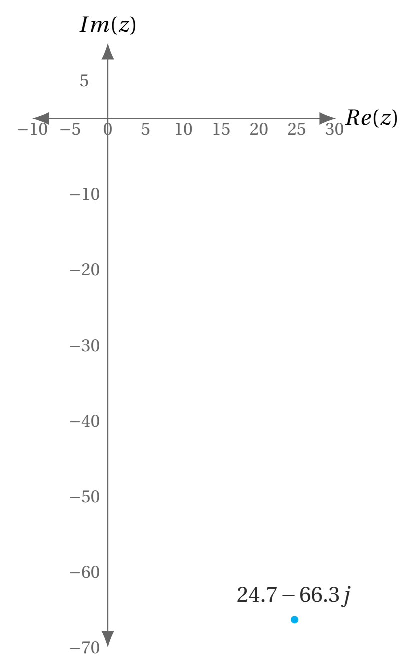


Figure 5.14

5.4 Exponential Form of a Complex Number

There is another method for representing complex numbers that involves the use of exponents and is called the **exponential form of a complex number**.

Exponential Form of a Complex Number

A complex number written in polar form can be represented in exponential form as

$$z = re^{j\theta}$$

where θ is in radians and e is the natural base.

While θ can be anything, all examples will restrict θ to $0 \leq \theta < 2\pi$.

Example 5.4.1: – Polar to exponential form

Write the complex number $5(\cos 90° + j \sin 90°)$ in exponential form.

Solution:

Since $5(\cos 90° + j \sin 90°)$ is already in polar form where $r = 5$ and $\theta = 90°$. Then all thats needed is to express θ in radians as required for the exponential form.

$$90° \cdot \frac{\pi}{180°} = \frac{\pi}{2}$$

Thus $5(\cos 90° + j \sin 90°) = 5e^{\frac{\pi}{2}j}$. Note, $\frac{\pi}{2}j$ is the exponent in this expression.

Example 5.4.2: – Rectangular to exponential form

Express $3 - 4j$ in expontial form.

Solution:

This example is in rectangular form where r and θ must be determined.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{3^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

and

$$\begin{aligned}\theta &= \tan^{-1}\frac{b}{a}\\ &= \tan^{-1}\frac{-4}{3}\\ &\approx -.93\\ &= -.93+2\pi && \text{add } 2\pi \text{ to get a result in the domain } 0 \le \theta < 2\pi\text{. This step is not necessary.}\\ &= 5.35\end{aligned}$$

Since we had to approximate θ by rounding off what was given by the calculator then the entire conversion from rectangular to exponential form (in this case) is an approximation as well. Thus we have,

$$3-4j \approx 5e^{5.35\,j}$$

Example 5.4.3: – Polar to exponential form

Express $15\underline{/135°}$ in exponential form.

Solution:

Since $15\underline{/135°}$ is an abbreviated form of the polar form, all that's needed is to convert 135° to radians.

$$135° \cdot \frac{\pi}{180°} = \frac{3\pi}{4}$$

Thus, $15\underline{/135°} = 15e^{\frac{3\pi}{4}\cdot j}$.

Example 5.4.4: – Rectangular to exponential form

Express $-3-4j$ in exponential form.

Solution:

This example is similar to Example 5.4.2 except the calculator will interpret θ to be in quadrant 1 since it's first operation will be to cancel out the division of the two negatives. We must be aware of this and add the equivalent of 180°, or in our case π radians, to the calculators result since we know that $-3-4j$ is in quadrant 3.

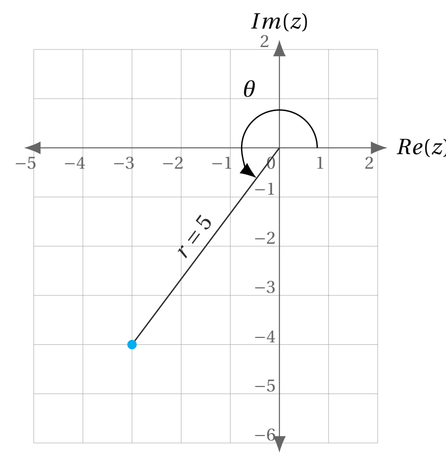


Figure 5.15

$$\begin{aligned}r &= \sqrt{a^2+b^2}\\ &= \sqrt{(-3)^2+(-4)^2}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

and

$$\theta = \tan^{-1}\frac{b}{a}$$
$$= \tan^{-1}\frac{-4}{-3}$$
$$\approx .93$$

This is the result from the calculator, but π radians must be added to get the correct angle.

$$= .93 + \pi$$
$$= 4.07$$

Thus, $-3 - 4i \approx 5e^{4.07i}$.

5.4.1 Multiplication and Division in Exponential Form

An advantage of using the exponential form of a complex number is that the basic properties of exponents obey the laws of exponents introduced in section 1.3.1. Specifically, there are three properties of interest: multiplication, division, and powers.

For convenience, these exponential properties are listed below:

$$a^n a^m = a^{n+m} \tag{5.5}$$

$$\frac{a^n}{a^m} = a^{n-m} \tag{5.6}$$

$$\left(a^n\right)^m = a^{nm} \tag{5.7}$$

$$(ab)^n = a^n b^n \tag{5.8}$$

Given, or determined, two complex numbers in exponential form, say $z_1 = r_1 e^{\theta_1 j}$ and $z_2 = r_2 e^{\theta_2 j}$, then we can multiply or divide them using the properties of exponents just described above. For example,

$$z_1 z_2 = \left(z_1 = r_1 e^{\theta_1 j}\right)\left(z_2 = r_2 e^{\theta_2 j}\right)$$
$$= r_1 r_2 e^{(\theta_1+\theta_2) j}$$

$$and \qquad \frac{z_1}{z_2} = \frac{r_1 e^{\theta_1 j}}{r_2 e^{\theta_2 j}}$$
$$= \frac{r_1}{r_2} e^{(\theta_1-\theta_2) j}$$

Example 5.4.5: – Multiply in exponential form

Multiply $4e^{\pi j}$ and $2e^{\frac{\pi}{2}j}$.

Solution:

$$\begin{aligned}\left(4e^{\pi j}\right)\left(2e^{\frac{\pi}{2}j}\right) &= 4(2)e^{\left(\pi+\frac{\pi}{2}\right)j} && \text{add the exponents}\\ &= 8e^{\frac{3\pi}{2}j}\end{aligned}$$

Example 5.4.6: – Divide in exponential form

Divide $4e^{\pi j}$ and $2e^{\frac{\pi}{2}j}$.

Solution:

$$\begin{aligned}\frac{4e^{\pi j}}{2e^{\frac{\pi}{2}j}} &= \frac{4}{2}e^{\left(\pi-\frac{\pi}{2}\right)j} && \text{subtract the exponents}\\ &= 2e^{\frac{\pi}{2}j}\end{aligned}$$

Appendix A

Direction - Headings and Bearings

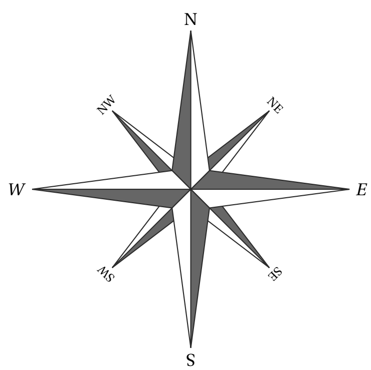


Figure A.1: Cardinal Rose

A.0.1 Heading

Many applications in technical mathematics involve vehicles, such as vessels and aircraft that navigate with respect to **cardinal direction**, or **cardinal points** which refer to north, south, east, and west. These cardinal directions are typically abreviated by N, S, E, and W.

A vehicles **heading** is the direction in which the *nose* is pointing. Headings are usually referenced either by a magnetic compass, or by instruments that reference the *lines of meridian* (true north and south lines). In either case, both of these types of instruments measure angles beginning off the north axis where a positive angle is measured in the clockwise direction as shown in figure A.2.

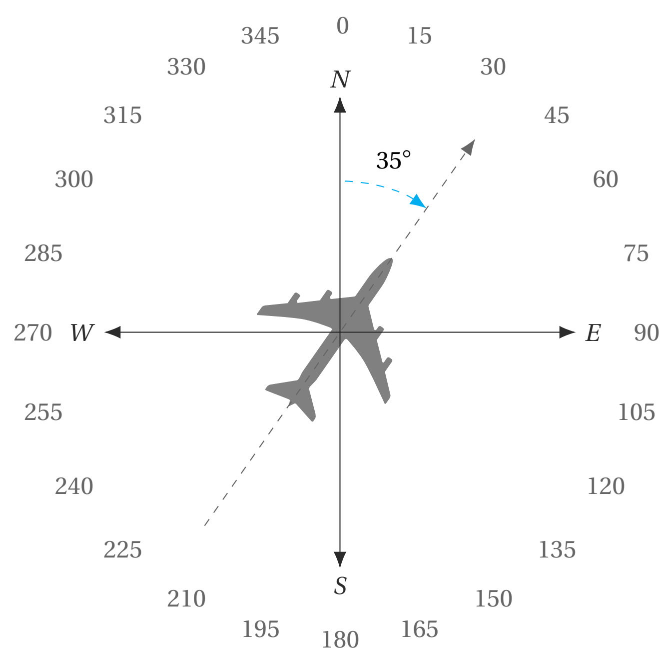


Figure A.2: Heading

A vehicles heading is constantly changing due to either crosswinds or currents. Thus a vehicles heading is not necessarily the direction the vehicle is traveling. Even if a vehicle maintains a specific heading its path of travel will not be straight since the instruments reference a compass that's directed north of a sphere. For instance, if a plane flies with a heading due north and does not change direction, then its heading will change from due north to due south once it has crossed the north pole.

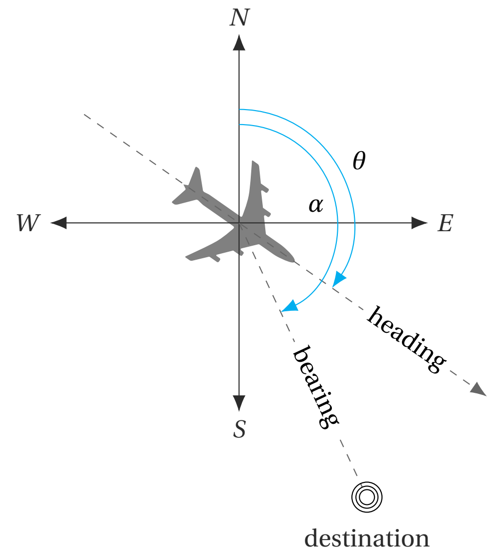


Figure A.3: Heading and Bearing

Note:
Without any crosswind or current, then heading and bearing would be the same direction. In the physical world this would almost never occur; however, in math (unless otherwise stated) the two terms are interchangeable.

In math, the heading to a specific location is stated as the number of degrees east or west of north or south. This means that the angles can be measured in a positive direction either clockwise or counterclockwise *of* either the north axis or the south axis from where the angle originates. See Figure A.4 for an illustration.

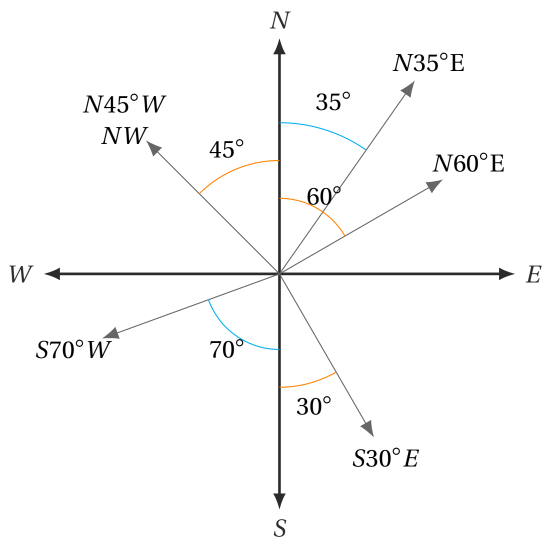


Figure A.4: Heading

A.0.2 Bearing

As stated before, heading is not necessarily the direction in which the vehicle is moving usually because of crosswinds or currents. **Bearing** is the angle (clockwise) between the North axis and the direction to the destination. Bearings are used to reference one point relative to another point where either of these points can be a static location(s) or a vehicle(s). Figure A.5 illustrates the bearing from one location with respect to another (either to or from).

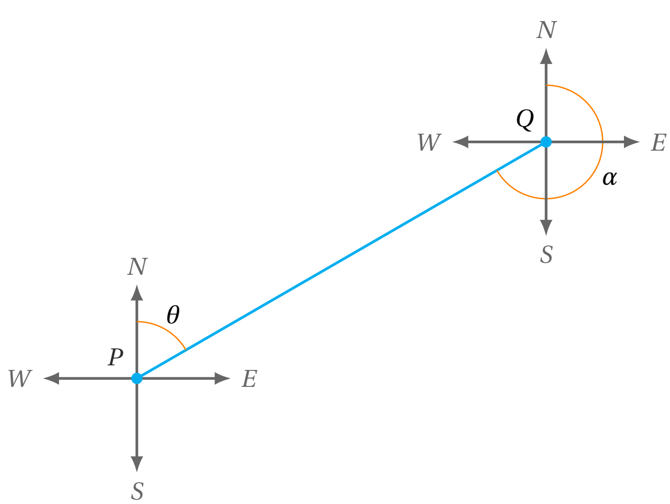


Figure A.5

In the above figure, the bearing for P to/from Q is θ, and the bearing for Q to/from P is α.
With regard to the differences between heading and bearing, I find it easiest to think of it this way. Regardless of the direction the vehicle is pointing (heading), the bearing is ultimately the direction the vehicle is traveling.

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