3.1 Introduction to Second Order Linear Equations

Second order differential equation: $\dfrac{d^2y}{dt^2} = f\left(t, y, \dfrac{dy}{dt}\right)$

Linear: $f\left(t, y, \dfrac{dy}{dt}\right) = g(t) - p(t)\dfrac{dy}{dt} - q(t)y$

In general we write:

$$y'' + p(t)y' + q(t)y = g(t)$$

Or if we have an initial value problem:

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \; y'(t_0) = y_0', \quad a < t < b$$

Note: $g(t)$, $p(t)$ and $q(t)$ are continuous functions of t on $a < t < b$.
There are two types: homogeneous if $G(t) = 0$ and nonhomogeneous if $G(t) \neq 0$.

We will begin modeling the behavior of a simple physical system: The bobbing motion of a floating object.

The buoyant force on an object in liquid is equal to the weight of the displaced liquid. So in its rest or equilibrium state, a floating object is subjected to two equal and opposite forces (1) its weight and (2) the weight of the liquid.

Consider the objects shown in figure 4.1 (a). The object has a uniform mass density ρ, constant cross sectional area A, and height L. The density of the liquid is ρ_l. If we wish to model the equilibrium solution we set

the weight of the liquid = the weight of the object

$$\rho_l AYg = \rho ALg \qquad \textbf{(1)}$$

Or

$$Y = \frac{\rho}{\rho_l} L$$

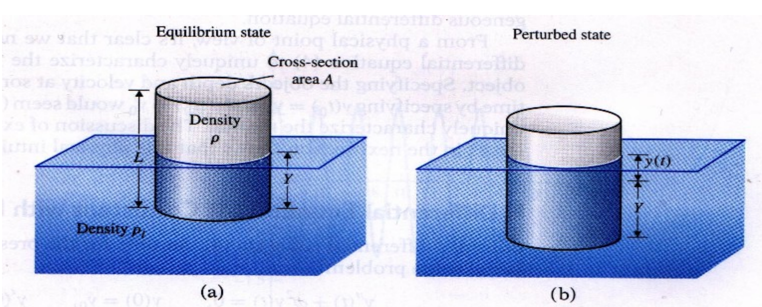

FIGURE 4.1
(a) The floating object is in its equilibrium or rest state when the weight of the displaced liquid is equal to the weight of the object.
(b) The object is in a perturbed state when it is displaced from its equilibrium position. At any time t, the quantity $y(t)$ measures how far the object is from its equilibrium position.

We would like to write an equation that models the motion of the object if it is given some displacement $y(t)$. Newton's law of motion says $\sum F = ma$.

We can use this idea because we can write equations for the forces acting on the object:

(a) The weight of the object: ρALg
(b) The buoyant force of the liquid $\rho_l Ag(Y + y(t))$

And the acceleration is the second derivative of the position $\frac{d^2}{dt^2}\left(Y + y(t)\right)$ and the mass is ρAL so we can write

$$\rho AL\frac{d^2}{dt^2}(Y + y(t)) = \rho ALg - \rho_l A(Y + y(t))g$$

And substituting equation **(1)** and taking the derivative on the left we get:

$$\rho ALy''(t) = -\rho_l Ay(t)g$$

Which we will write as

$$y''(t) + \omega^2 y(t) = 0 \quad \text{where } \omega^2 = \frac{\rho_l g}{\rho L}$$

We will also apply the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$.

It turns out (we will see why later) that the solution to this equation is

$$y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$$

where the constants are determined by the initial conditions. Solving for the constants gives us that $C_1 = y_0'/\omega$ and $C_2 = y_0$ so the solution is

$$\boxed{y(t) = \frac{y_0'}{\omega}\sin(\omega t) + y_0 \cos(\omega t)} \text{ where } \boxed{\omega^2 = \frac{\rho_l g}{\rho L}}$$

Ex 1: A cylindrical block of wood has a circular cross-sectional area. The diameter of the base is 1 ft and the height is 2 ft. The wood is hard oak, which weighs 50 lb/ft^3. The block is initially floating at rest in water. The mass of water is 1 g/cm^3 and that 1 ft^3 of water weighs 62.4 lb. Suppose that the block is perturbed from its rest state by giving it an initial downward velocity $y'(0) = y_0'$. It is observed that the block, in its subsequent bobbing motion , sinks into the water to the point where it just becomes totally submerged. What was the initial downward velocity y_0'?

Theorem 1: Consider the initial value problem

$$y''+p(t)y'+q(t)y = g(t), \quad y(t_0)=y_0, \quad y'(t_0)=y_0'$$

where $p(t)$, $q(t)$ and $g(t)$ are continuous on I. Then there is **exactly one solution** $y=\phi(t)$ and the **solution exists throughout *I*.**

Ex 2: Find the largest interval in which $(t-1)y''-3ty'+4y=\sin t, \quad y(-2)=2, \quad y'(-2)=1$ is certain to have a unique, twice differentiable solution.

Step 1: Make the IVP look like the formula in Theorem 1.

$$y''-\frac{3t}{(t-1)}y'+\frac{4}{(t-1)}y=\frac{\sin t}{(t-1)}$$

Step 2: Find where $p(t)$, $q(t)$ and $g(t)$ are continuous.

Step 3: Choose the appropriate interval which includes the initial t_0

Ex 3: Find the largest interval in which $y''+(\cos t)y'+(3\ln|t|)y=0, \quad y(2)=3, \quad y'(2)=1$ is certain to have a unique, twice differentiable solution.