Contents

3 Second and Higher Order Differential Equations

3.2 The General Solutions of Homogeneous Equations

We will start by looking at solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continuous on (a, b).

Definition: Let $f_1(t)$ and $f_2(t)$ be any two functions having a common domain, and let c_1 and c_2 be any two constants. Then the function $F(t) = c_1f_1(t) + c_2f_2(t)$ is a linear combination of $f_1(t)$ and $f_2(t)$. We can extend the definition in the obvious way to describe any number of functions.

Principle of Superposition

Theorem 3.2: The principle of superposition:
If $y_1(t)$ and $y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$ defined on the interval $a < t < b$, where $p(t)$ and $q(t)$ are continuous on (a, b). Then the linear combination $y(t) = C_1y_1(t) + C_2y_2(t)$ is also a solution of the differential equation.

Proof For simplicity we will assume $C_1 = C_2 = 1$ but if you include those the proof is the same.

$$\begin{aligned} y &= y_1 + y_2 \\ y' &= y_1' + y_2' \\ y'' &= y_1'' + y_2'' \end{aligned} .$$

Substitute back into $y'' + p(t)y' + q(t)y = 0$ to get:

$$y_1'' + y_2'' + p(t)(y_1' + y_2') + q(t)(y_1 + y_2) = \underbrace{y_1'' + p(t)y_1' + q(t)y_1}_{=0} + \underbrace{y_2'' + p(t)y_2' + q(t)y_2}_{=0} = 0$$

QED

Definition 3.1. If $y_1(t)$ and $y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$ and every other solution $y(t)$ can be written as a linear combination of these two (ie. $y(t) = C_1y_1(t)+C_2y_2(t)$) then $y_1(t)$ and $y_2(t)$ are a **fundamental set of solutions**.

Consider $y'' + p(t)y' + q(t)y = 0$ with the initial conditions $y(t_0) = y_0$, $y'(t_0) = y'_0$. If we start with 2 solutions y_1 and y_2 then $y = C_1 y_1 + c_2 y_2$ is also a solution and we can solve for the constants C_1 and C_2 by using the initial conditions. We get a system of equations:

$$\begin{aligned} y_0 &= C_1 y_1(t_0) + C_2 y_2(t_0) \\ y'_0 &= C_1 y'_1(t_0) + C_2 y'_2(t_0) \end{aligned}$$

We can use Cramer's Rule to solve the system:

$$C_1 = \frac{\det \begin{bmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{bmatrix}}{\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{bmatrix}} \quad \text{and} \quad C_2 = \frac{\det \begin{bmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{bmatrix}}{\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{bmatrix}}$$

Notice that C_1 and C_2 have solutions as long as

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{bmatrix} \neq 0$$

or $y_1(t_0)y'_2(t_0) - y_2(t_0)y'_1(t_0) \neq 0$

This determinant is known as the **Wronskian:** $W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix} \neq 0$

Fundamental Solutions

Theorem: Suppose $y_1(t)$ and $y_2(t)$ are two solutions to

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b,$$

where $p(t)$ and $q(t)$ are continuous on (a, b). Let $W(t)$ be the Wronskian of $y_1(t)$ and $y_2(t)$. If there is a point t_0 in (a, b) such that $W(t_0) \neq 0$, then $\{y_1(t), y_2(t)\}$ is a **fundamental set of solutions** for $y'' + p(t)y' + q(t)y = 0$.

Abel's Theorem

Abel's Theorem: Suppose $y_1(t)$ and $y_2(t)$ are two solutions to

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b,$$

where $p(t)$ and $q(t)$ are continuous on (a, b). Let $W(t)$ be the Wronskian of $y_1(t)$ and $y_2(t)$. If t_0 is any point in (a, b), then

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}$$

Why do we care about Abel's Theorem? Because it says that if the Wronskian is not zero at any point of (a, b) then it is not zero at every point of (a, b).

Example 3.2.1.

(a) Determine whether the given functions are solutions of the differential equation.

(b) If both functions are solutions, calculate the Wronskian. Does this calculation show that the two functions form a fundamental set of solutions?

(c) If the two functions have been shown in part (b) to form a fundamental set, construct the general solution and determine the unique solution satisfying the given initial conditions.

1. $y'' + y = 0$; $y_1(t) = \sin t \cos t, \quad y_2(t) = \sin t; \qquad y\left(\frac{\pi}{2}\right) = 1$ and $y'\left(\frac{\pi}{2}\right) = 1$

2. $y'' - 4y' + 4y = 0$; $y_1(t) = e^{2t}, \quad y_2(t) = te^{2t}; \qquad y(0) = 2$ and $y'(0) = 0$

3. $ty' + y = 0,\quad 0 < t < \infty$; $y_1(t) = \ln t,\quad y_2(t) = \ln(3t)$; $\qquad y(3) = 0$ and $y'(3) = 3$

4. $4y'' + y = 0$; $y_1(t) = \sin(t/2 + \pi/3),\quad y_2(t) = \sin(t/2 - \pi/3)$; $\qquad y(0) = 0$ and $y'(0) = 1$

3.3 Constant Coefficient Differential Equations

We will begin with linear, homogeneous, 2^{nd} order differential equation with constant coefficients. This means an equation of the form:

$$ay'' + by' + cy = 0 \qquad \text{where } a,\ b,\ \text{and } c \text{ are constants.}$$

If we start with something like $y'' - y = 0$ then we have two obvious solutions:

$$\begin{aligned} y_1 &= C_1 e^{t} \\ y_2 &= C_2 e^{-t} \end{aligned}$$

Now consider the more general case $ay'' + by' + cy = 0$. A solution to this equation could be of the form

$$y = e^{rt}.$$

To see that this is a solution we will take two derivatives and substitute back into the original equation:

$$\begin{aligned} y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}.$$

So if we substitute back into $ay'' + by' + cy = 0$ to solve for r we get:

$$(ar^2 + br + c)e^{rt} = 0.$$

$$\boxed{ar^2 + br + c = 0 \text{ is called the } \textbf{characteristic equation}.}$$

The solutions to the characteristic equation are the exponents for $y = e^{rt}$. There are always 2 solutions to a quadratic equation so if r_1 and r_2 are distinct real solutions to $ar^2+br+c = 0$ then

$$\boxed{y = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

where C_1 and C_2 are arbitrary constants, is the solution to $ay'' + by' + cy = 0$.
If r_1 and r_2 are repeated real solutions or complex solutions then we will deal with that later.

Example 3.3.1. Solve $y'' + 2y' - 3y = 0$

Step 1: Solve the characteristic equation.

$$r^2 + 2r - 3 = 0$$

Step 2: Write down the solution. $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$$\boxed{y = C_1 e^{-3t} + C_2 e^{t}}$$

Example 3.3.2. Initial Value Problem: Solve the equation and describe the behavior as $t \to \infty$.

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, \qquad y'(0) = 0$$

Step 1: Solve the characteristic equation.

$$6r^2 - 5r + 1 = 0$$

Step 2: Write the solution: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Step 3: Solve for the constants C_1 and C_2

Example 3.3.3. Solve the differential equation and describe the behavior as $t \to \infty$.

$$2y'' + y' - 4y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$$

3.4 Repeated Roots and Reduction of Order

Recall: 2nd order differential equation $ay'' + by' + cy = 0$ has characteristic equation:

$$ar^2 + br + c = 0$$

with solutions

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = r_1 \text{ and } r_2.$$

There are three cases for these roots:

Case 1: $b^2 - 4ac > 0$ gives two distinct real roots r_1 and r_2

The solutions to $ay'' + by' + cy = 0$ are $y = C_1 e^{r_1 t}$ and $y_2 = C_2 e^{r_2 t}$. You can put them together to get the general solution

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case 2: $b^2 - 4ac < 0$ gives two complex conjugate solutions $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$

The solutions to $ay'' + by' + cy = 0$ are discussed in section 3.5.

Case 3: $b^2 - 4ac = 0$ gives 1 repeated root r.

The two solutions to $ay'' + by' + cy = 0$ are $y_1 = C_1 e^{rt}$ and $y_2 =?$

If we know one solution to a differential equation we can find a second solution using a technique known as Reduction of Order.

Suppose we know one solution y_1 to the equation $y'' + p(t)y' + q(t)y = 0$ then to find a linearly independent second solution we can use a nonconstant multiple of our original solution.

Let

$$y_2(t) = v(t)y_1(t).$$

Or if you don't want to use as much the function notation you can abbreviate it as

$$y_2 = v(t)y_1.$$

Since we want this to be a solution to the equation $y'' + p(t)y' + q(t)y = 0$ we will need two derivatives:

$y_2' = v(t)y_1' + v'(t)y_1$

$y_2'' = v(t)y_1'' + v'(t)y_1' + v'(t)y_1' + v''(t)y_1$

We will substitute y_2, y_2' and y_2'' back into the original equation to get:

$$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

$$v(t)y_1'' + 2v'(t)y_1' + v''(t)y_1 + p(t)\left(v(t)y_1' + v'(t)y_1\right) + q(t)\left(v(t)y_1\right) = 0$$

Simplify and combine the terms by $v''(t)$, $v'(t)$ and $v(t)$ we see that

$$v''(t)y_1 + \left(2y_1' + p(t)y_1\right)v'(t) + \left(y_1'' + p(t)y_1' + q(t)y_1\right)v(t) = 0.$$

This leaves us with the first order differential equation:

$$y_1 v''(t) + \left(2y_1' + p(t)y_1\right)v'(t) = 0$$

Which is simply a first order differential equation in $v'(t)$. We can solve this equation for $v'(t)$ using the techniques we learned in Ch 2 and then we can integrate to find $v(t)$. Then we have our second solution:

$$y_2 = v(t)y_1$$

Example 3.4.1. Find the general solution for $4y'' + 12y' + 9y = 0$

Start with the characteristic equation: $4r^2 + 12r + 9 = 0$

Using this r we can write our solutions: $y_1 = C_1e^{-3t/2}$ and $y_2 = v(t)e^{-3t/2}$

Substitute y_2, y_2' and y_2'' back into the original equation

In general:

Repeated Roots Solutions

If r is the only solution to the characteristic equation then two linearly independent solutions to $ay'' + by' + cy = 0$ are

$$y_1 = C_1 e^{rt} \quad \text{and} \quad y_2 = C_2 t e^{rt}$$

Example 3.4.2. Find the general solution for $9y'' - 12y' + 4y = 0$, $y(0) = -1$, $y'(0) = 2$

Example 3.4.3. Suppose we know that one solution to $t^2 y'' + 3ty' + y = 0$ is $y_1 = t^{-1}$ find a second linearly independent solution using reduction of order.

3.5 Complex Roots of the Characteristic Equation

Recall:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned}$$

We will use these formulas to convert complex roots of $ay'' + by' + cy = 0$ to real solutions. If $r = \alpha \pm \beta i$ are the solutions to the characteristic equation them we know how to write the general solution:

$$y = C_1 e^{(\alpha+\beta i)t} + C_2 e^{(\alpha-\beta i)t}$$

but this is not a useful form. We want real solutions. We are going to use the identities at the top of the page to convert this into the form

$$y = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right).$$

$e^{(\alpha+\beta i)t} = e^{\alpha t}e^{\beta i t} = e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)$
$e^{(\alpha-\beta i)t} = e^{\alpha t}e^{-\beta i t} = e^{\alpha t}\left(\cos\beta t - i\sin\beta t\right)$

So the general solution is

$$y = C_1 e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right) + C_2 e^{\alpha t}\left(\cos\beta t - i\sin\beta t\right)$$

which simplifies to

$$y = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right).$$

Example 3.5.1. Find the general solution for $y'' + 2y' + 2y = 0$

Step 1: Characteristic equation

Step 2: Write the solution. Here α = -1 and β = 1

Example 3.5.2. Find the general solution for

$$y'' + 4y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$$

Example 3.5.3. Find the general solution for

$$y'' - 2y' + 5y = 0, \qquad y\left(\frac{\pi}{2}\right) = 0, \qquad y'\left(\frac{\pi}{2}\right) = 2$$

We would like to be able to write the solution

$$y = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t \tag{3.1}$$

as one trigonometric function of the form:

$$y(t) = Re^{\alpha t}\cos(\beta t - \delta)$$

Where$Re^{\alpha t}$ is the amplitude and δ is the phase angle. Using a trigonometric identity we can expand this to be:

$$y = Re^{\alpha t}\cos\delta\cos\beta t + Re^{\alpha t}\sin\delta\sin\beta t \tag{3.2}$$

Set the two equations (3.1) and (3.2) equal to each other.

$$Re^{\alpha t}\cos\delta\cos\beta t + Re^{\alpha t}\sin\delta\sin\beta t = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$$

$e^{\alpha t}$ cancels on both sides and we can solve for R and δ with the two equations:

$$R\cos\delta = A \qquad \text{and} \qquad R\sin\delta = B$$

square both and add them together to get

$$R^2\cos^2\delta + R^2\sin^2\delta = A^2 + B^2$$

So $\boxed{R = \sqrt{A^2 + B^2}}$

To solve for δ you divide the equations to get $\boxed{\tan\delta = \dfrac{B}{A}}$

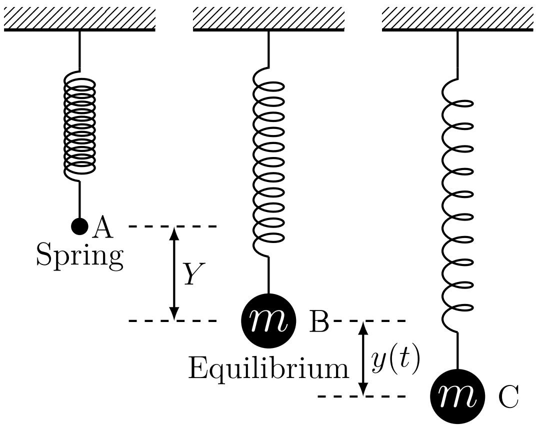
Be sure to pick the correct δ since there are two choices.

Example 3.5.4. Write this equation in the form $y(t) = Re^{\alpha t}\cos(\beta t - \delta)$

$y = -e^{-t}\cos t + \sqrt{3}e^{-t}\sin t$

3.6 Unforced Mechanical Vibrations

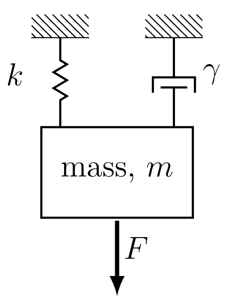
Consider the spring and mass system shown here:



Y is the elongation of the spring in the downward direction caused by the mass m. $y(t)$ is the distance traveled by the mass from the equilibrium position. The spring constant is k. If we do not know the value of k we can solve for it. From physics we know that the force of the spring is nearly proportional to Y. So at equilibrium we can write the following equations to solve for k:

$$\begin{aligned} F &= kY \\ mg &= kY \\ k &= \frac{mg}{Y} \end{aligned}$$

In reality all systems have some amount of damping so the usual spring-mass-damper system can be modeled as follows: k is the spring constant, γ is the damping constant, and m is the mass.



From our free body diagram we can find the forces acting on the mass:

Force of the spring	$k[Y + y(t)]$	$\uparrow$
Force of gravity	mg	$\downarrow$
Force of damper	$\gamma\, y'(t)$	$\uparrow$
Some external forcing function:	$F(t)$	

Using the standard formula $\sum F = ma = mv''(t)$ to get a differential equation:

$$\begin{aligned} mg - k[Y + y(t)] - \gamma y'(t) + F(t) &= my''(t) \\ my''(t) + \gamma y'(t) + ky(t) &= F(t) \qquad (3.3) \end{aligned}$$

Units: The units all have to be force units so

$$k = \frac{\text{force}}{\text{displacement}} = \frac{kg}{s^2} \qquad \text{force:} \quad \text{lbs, N, } \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$$

$$\text{displacement:} \quad \text{m, ft, etc.}$$

$$\gamma = \frac{\text{force}}{\text{speed}} = \frac{kg}{s} \qquad \text{speed:} \quad \frac{\text{m}}{\text{s}}, \frac{\text{ft}}{\text{s}}, \text{ etc.}$$

The characteristic equation for 3.3

$$mr^2 + \gamma r + k = 0$$

has solutions

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

There are 3 possible outcomes:

$\gamma^2 - 4mk < 0$:	Complex solutions, underdamped, oscillation.
$\gamma^2 - 4mk = 0$:	One repeated solution, critically damped, no oscillation.
$\gamma^2 - 4mk > 0$:	Two negative real solutions, overdamped, no oscillation.

Example 3.6.1. A mass of 100 g stretches a spring 5 cm. If the mass is pulled down 2 cm, given a downward velocity of 10 cm/sec, and if there is no damping, determine the position $y(t)$ of the mass at any time t. Find the frequency, period, and amplitude of the motion.

Step 1: Draw a picture and identify what you know including Initial Conditions:

Step 2: Write down a useful equation(s):

$$my'' + ky = 0$$

Step 3: Solve:

$$y(t) = 2\cos(14t) + \frac{5}{7}\sin(14t) \tag{3.4}$$

We would like to be able to write equation 3.4 as one trigonometric function of the form:

$$y(t) = R\cos(\mu t - \delta)$$

Where R is the amplitude or maximum displacement of our vibration, μ is the natural frequency (Hz) of the vibration and δ is the phase angle. If we expand that with the cosine angle difference identity ($\cos(A - B) = \cos A \cos B + \sin A \sin B$) what we get is something of the form:

$$y(t) = R\cos\delta\cos(\mu t) + R\sin\delta\sin(\mu t) \tag{3.5}$$

We can compare the identity (equation 3.5) to the answer (equation 3.4).

$$R\cos\delta\cos(\mu t) + R\sin\delta\sin(\mu t) = 2\cos(14t) + \frac{5}{7}\sin(14t)$$

and solve for R and δ. Assuming that $\mu = 14$ we can compare the coefficients:

$$R\cos\delta = 2 \qquad \text{and} \qquad R\sin\delta = \frac{5}{7}$$

Then

$$R^2\cos^2\delta + R^2\sin^2\delta = R^2 = 2^2 + \left(\tfrac{5}{7}\right)^2 \implies \boxed{R = \frac{\sqrt{221}}{7}}$$

δ can be found by taking the quotient of the two equations:

$$\frac{R\sin\delta}{R\cos\delta} = \tan\delta = \frac{5}{14} \implies \boxed{\delta = \tan^{-1}\left(\tfrac{5}{14}\right)}$$

The final solution is

$$\boxed{y(t) = \frac{\sqrt{221}}{7}\cos(14t - \delta)}$$

Example 3.6.2. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/sec, find its position y at any time t . Plot y versus t . Determine the quasi frequency and the quasi period.

Example 3.6.3. A mass weighing 8 pounds stretches a spring 6 inches before coming to rest. It is pulled down three more inches before being released with an initial velocity of 1 foot per second. Find the amplitude, period, and circular frequency of the resulting motion.

Example 3.6.4. An object weighing 32 pounds stretches a spring 2 feet. It is pulled down 6 inches and released in a medium where resistance is 4 times the velocity. Describe the motion of the spring.

Example 3.6.5. The resistance factor in **Example** 3.6.4 is doubled to 8 times the velocity. Describe the resulting motion.

Example 3.6.6. The resistance in **Example** 3.6.4 is increased to 10 times velocity. Describe the motion.

3.7 The General Solution of a Linear Nonhomogenous Equation

Given the differential equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{3.6}$$

we know that this is nonhomogeneous if $g(t) \neq 0$. For every nonhomogeneous equation there is a corresponding homogenous equation:

$$y'' + p(t)y' + q(t)y = 0 \tag{3.7}$$

Suppose that Y_1 and Y_2 are two solutions to the nonhomogenous equation 3.6 and $\{y_1, y_2\}$ are a fundamental set of solutions for the corresponding homogenous equation 3.7, then:

$$Y_1 - Y_2 = C_1 y_1 + C_2 y_2$$

Translation: There is only **ONE** particular solution to any nonhomogeneous differential equation.

The General Solution

The complete solution to equation 3.6 $y'' + p(t)y' + q(t)y = g(t)$ can be written as:

$$y = C_1 y_1 + C_2 y_2 + Y_p$$

Where $\{y_1, y_2\}$ are a fundamental set of solutions for the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$, C_1 and C_2 are arbitrary constants, and Y_p is some particular solution to equation 3.6

The Principle of Superposition

Let $u(t)$ be a solution of $y'' + p(t)y' + q(t)y = g_1(t)$ and $v(t)$ be a solution to $y'' + p(t)y' + q(t)y = g_2(t)$. If a_1 and a_2 are any constants then the function

$$y_p = a_1 u(t) + a_2 v(t)$$

is a solution to

$$y'' + p(t)y' + q(t)y = a_1 g_1(t) + a_2 g_2(t)$$

3.8 The Method of Undetermined Coefficients

The solution to every differential equation $y'' + p(t)y' + q(t)y = g(t)$ is always of the form

$$y = y_h + y_p$$

where y_h is the homogeneous solution and y_p is the particular solution.
There are always 3 steps to solving $y'' + p(t)y' + q(t)y = g(t)$

Step 1: Find the homogeneous solution $y_h = C_1y_1 + C_2y_2$.

Step 2: Find the particular solution y_p.

Step 3: Add them together $y = y_h + y_p = C_1y_1 + C_2y_2 + y_p$

We know how to do Step 1: Solve the characteristic equation.
We know how to do Step 3.
How are we going to find the particular solution?
Answer: Guess a solution that looks like your answer $g(t)$.
This is known as the **Method of Undetermined Coefficients**.

Example 3.8.1. $y'' - 2y' - 3y = e^{-3t}$

Example 3.8.2. $y'' + 2y' = 3 + 4\sin(2t)$

Example 3.8.3. $y'' + 2y' + y = 2e^{-t}$

The initial guess for the method of undetermined coefficients to solve the differential equation

$$ay'' + by' + cy = g_i(t)$$

is summarized in the following table. The first column is the form of $g_i(t)$ and the second column is the form of the particular solution $Y_i(t)$. Notice that each equation has t^s. Choose s to be the smallest integer such that NO term of $Y_i(t)$ is a solution to the homogeneous equation.

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_n t^n + \cdots + a_1 t + a_0$	$t^s \left[A_n t^n + \cdots + A_1 t + A_0\right]$
$P_n(t)e^{\alpha t}$	$t^s \left[A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\right] e^{\alpha t}$
$P_n(t)e^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$t^s \left[A_n t^n + \cdots + A_1 t + A_0\right] e^{\alpha t}\sin(\beta t) +$ $+ t^s \left[B_n t^n + \cdots + B_1 t + B_0\right] e^{\alpha t}\cos(\beta t)$

Example 3.8.4. Determine a suitable form for the particular solution $Y(t)$ if the method of undetermined coefficients is to be used.

$$y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin(2t)$$

3.9 Variation of Parameters: (The plow method)

There are always 3 steps to solving the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Step 1: Find the homogeneous solution. $y_h = C_1 y_1 + C_2 y_2$.

Step 2: Find the particular solution: y_p

Step 3: Add them together: $y = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$

In section 3.8 we saw that we could guess at a solution that looked like the answer. That is fine if your answer is nice but it doesn't always work well. Variation of parameters is a completely general form that applies to all situations. However, it isn't always possible to solve the problem explicitly because in the end there are always integrals to be evaluated.

<u>Variation of Parameters:</u>

Start with $y'' + p(t)y' + q(t)y = g(t)$

Step 1: Solve the homogeneous equation for the family of solutions y_1, y_2

Step 2: Let the particular solution have the form:

$$y_p = u_1(t)y_1 + u_2(t)y_2$$

where $u(t)$ and $v(t)$ are functions that we will determine. To solve for u_1 and u_2 we need to find y_p' and y_p'' and substitute back into the original equation. Don't forget the product rule.

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

Now at this point we have generated 4 terms and 4 unknown values (u_1, u_2, u_1' and u_2') so we need to put some constraints on the system. There are many choices we could make here but the best choice is to assume that

$$u_1' y_1 + u_2' y_2 = 0. \tag{3.8}$$

This equation will be one that we will use to solve for u_1 and u_2 and with this constraint the derivative simplifies to $y_p' = u_1 y_1' + u_2 y_2'$. Now take a second derivative

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

Now put y_p, y_p' and y_p'' back into the original differential equation and simplify:

$$\begin{aligned}
g(t) &= y_p'' + p(t)y_p' + q(t)y_p \\
&= u_1\, y_1'' + u_1'\, y_1' + u_2\, y_2'' + u_2'\, y_2' + p(t)[u_1\, y_1' + u_2\, y_2'] + q(t)[u_1\, y_1 + u_2\, y_2] \\
&= u_1\, y_1'' + p(t)u_1\, y_1' + q(t)u_1\, y_1 + u_2\, y_2'' + p(t)u_2\, y_2' + q(t)u_2\, y_2 + u_1'\, y_1' + u_2'\, y_2' \\
&= u_1\, (y_1'' + p(t)\, y_1' + q(t)\, y_1) + u_2\, (y_2'' + p(t)\, y_2' + q(t)\, y_2) + u_1'\, y_1' + u_2'\, y_2'
\end{aligned}$$

Which simplifies to

$$g(t) = u_1'\, y_1' + u_2'\, y_2' \tag{3.9}$$

No we can solve equations 3.8 and 3.9 for u_1 and u_2

$$\begin{aligned}
u_1'\, y_1 + u_2'\, y_2 &= 0 \\
u_1'\, y_1' + u_2'\, y_2' &= g(t)
\end{aligned}$$

Solving this system is straight forward. Solve the first for u_1' and plug that into the second and simplify a little as follows:

$$\begin{aligned}
u_1' &= -\frac{u_2' y_2}{y_1} \\
g(t) &= \left(-\frac{u_2' y_2}{y_1}\right) y_1' + u_2' y_2' \\
g(t) &= u_2' \left(y_2' - \frac{y_2 y_1'}{y_1}\right) \\
u_2'(t) &= \frac{y_1 g(t)}{y_1 y_2' - y_2 y_1'}
\end{aligned}$$

After plugging this back into u_1 we get

$$u_1' = -\frac{y_2 g(t)}{y_1 y_2' - y_2 y_1'}$$

Now, recall that if y_1, y_2 form a fundamental set of linearly independent solutions to the characteristic equation, then the Wronskian will not equal zero. So, finally we need to integrate u_1' and u_2'

$$u_1 = -\int \frac{y_2 g(t)}{y_1 y_2' - y_2 y_1'}, \qquad u_2 = \int \frac{y_1 g(t)}{y_1 y_2' - y_2 y_1'}$$

These results are summarized here:

Variation of Parameters

Consider the differential equation,

$$y'' + q(t)y' + r(t)y = g(t)$$

Assume that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for

$$y'' + q(t)y' + r(t)y = 0$$

Then the particular solution to the nonhomogeneous differential equation is

$$y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}$$

Note: The above equation makes use of **Cramer's Rule** and isn't completly necessary for obtaining the particular solution. I usually just solve the system by guessing the answer of the form $y_p = u(t)y_1 + v(t)y_2$ and then applying the constraint $u'y_1 + v'y_2 = 0$. From there I solve the system and integrate u_1' and u_2' at the end.

Example 3.9.1. Solve $y'' + 9y = 9\sec^2 3t$

Step 1: Solve homogenous equation $y'' + 9y = 0$

$$y_h = C_1 \cos 3t + C_2 \sin 3t$$

Example 3.9.2. Solve

$$ty'' - (1+t)y' + y = t^2e^{2t}$$

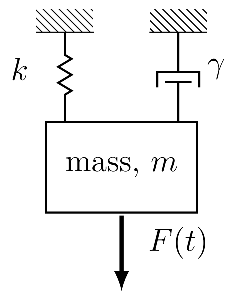
where the homogenous solutions are $y_1 = 1+t$ and $y_2 = e^t$.

Solution: For the particular solution we will guess:

$$y_p = v(1+t) + u(e^t)$$

3.10 Forced Mechanical Vibrations

The nonhomogeneous case of section 3.6



In these problems we have some external forcing function driving the system. The forcing function can be almost anything but usually it is periodic.

$$F(t) = F_1 \cos(\omega t) + F_2 \sin(\omega t)$$

So our equation looks like:

$$\begin{aligned} my''(t) + \gamma y'(t) + ky(t) &= F(t) \\ my''(t) + \gamma y'(t) + ky(t) &= F_1 \cos(\omega t) + F_2 \sin(\omega t) \end{aligned}$$

and the solution is once again

$$y(t) = y_h(t) + y_p(t) \tag{3.10}$$

where $y_h(t)$ is the homogeneous solution and $y_p(t)$ is the particular solution.

Two cases here:

I. If there is no damping (ie. $\gamma = 0$) then the solutions to 3.10 are of the form

$$y_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

$$y_p(t) = \begin{cases} A\cos(\omega t) + B\sin(\omega t) & \text{when } \omega \neq \omega_0 \\ At\cos(\omega t) + Bt\sin(\omega t) & \text{when } \omega = \omega_0 \end{cases} \tag{3.11}$$

The second case $\omega = \omega_0$ is known as resonance. When the frequency of the forcing function is the same as the natural frequency of the system then the motion of they system is unbounded as $t \to \infty$.

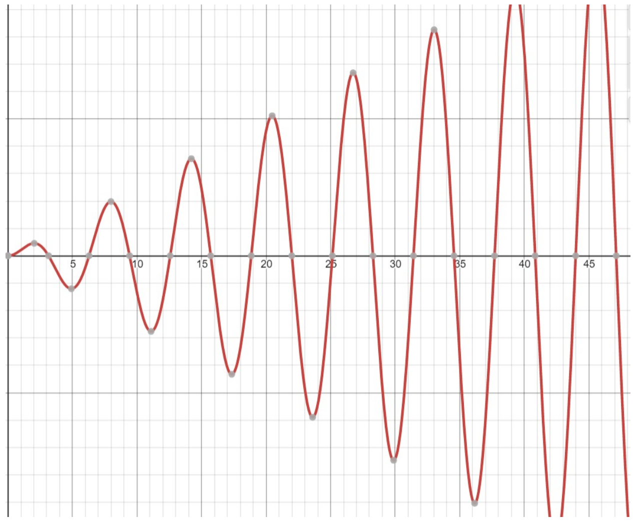


Figure 1: $y = 0.25t \sin t$

II. If γ is not zero then our solutions all have an e^{-rt} component in the homogenous solutions ($y_h(t)$). The total solution is

$$y(t) = y_h(t) + y_p(t)$$

and as $t \to \infty$ the homogeneous part, called the **transient solution**, goes to zero so in the long term you are only left with the particular solution. The particular solution is therefore called the **steady state solution** or **forced response**.

Example 3.10.1. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10 \sin\left(\frac{t}{2}\right)$ N and moves in a medium that imparts a viscous force of 2N when the speed is 4 cm/sec. If the mass is set in motion from its equilibrium position with an initial upward velocity of 3 cm/sec find an expression for the position of the mass at any time t. $y(t)$ is measured positive upwards. Identify the transient and steady state parts of the solution.

IMPORTANT: make sure your units match up.

3.11 Higher Order Linear Homogeneous Differential Equations

Existence and Uniqueness

Existence and Uniqueness

Let $p_0(t), p_1(t), \dots, p_{n-1}(t)$ and $g(t)$ be continuous functions defined on the interval $a < t < b$, and let t_0 be in (a, b). Then the initial value problem

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_2(t)y'' + p_1(t)y' + p_0(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad y''(t_0) = y_0'', \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

has a unique solution on the entire interval (a, b).

We need n initial conditions to solve an initial value problem (IVP) here. As in the 2$^{\text{nd}}$ order case we will find n linearly independent solutions to the homogeneous equation. We can still determine linear independence by calculating the Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $W \neq 0$ then $y_1, y_2, \dots, y_n$ are linearly independent and form a **fundamental set of solutions** for

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0.$$

For the nonhomogeneous case there is still only one particular solution y_p and the total solution is

$$Y(t) = y_h(t) + y_p(t).$$

Recall: A set of solutions is a **fundamental set of solutions** if every solution of the differential equation can be represented as a linear combination of the the elements of the set.

Abel's Theorem

Let $y_1(t), y_2(t), \ldots, y_n(t)$ be n solutions of the homogeneous linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0, \qquad a < t < b,$$

Where $p_0(t), p_1(t), \ldots, p_{n-1}(t)$ are continuous functions on (a, b). Let $W(t)$ be the Wronskian of $y_1(t), y_2(t), \ldots, y_n(t)$. If t_0 is any point in (a, b), then

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p_{n-1}(s)ds}, \qquad a < t < b$$

Why do we care about Abel's Theorem? Because it says that if the Wronskian is not zero at **any** point of (a, b) then it is not zero at **every** point of (a, b).

Definition 3.2. A set of functions defined on a common domain, say $f_1(t), f_2(t), \ldots, f_r(t)$ defined on the interval $a < t < b$, is called a **linearly dependent set** if there exist constants $k_1, k_2, \ldots, k_r$, not all zero, such that

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_r f_r(t), \qquad a < t < b.$$

A set of functions that is not linearly dependent is called **linearly independent**.

Linearly dependent essentially means that the functions $f_1(t), f_2(t), \ldots, f_r(t)$ are all different, while dependent sets are not really different.

Example 3.11.1. Solve the initial value problem given the following information.

$$y''' - y' = 0; \qquad y(0) = 4, \qquad y'(0) = 1, \qquad y''(0) = 3$$
$$y_1(t) = 1, \qquad y_2(t) = e^t, \qquad y_3(t) = e^{-t}$$

Example 3.11.2. Consider the differential equation $y'' + 2ty' + t^2y = 0$ on the interval $-\infty < t < \infty$. Assume that y_1 and y_2 are two solutions satisfying the given initial conditions.

(a) Do the solutions form a fundamental set?

(b) Do the two solutions form a linearly independent set of functions on $-\infty < t < \infty$.

1. $y_1(1) = 2, \qquad y_1'(1) = 2, \qquad y_2(1) = -1, \qquad y_2'(1) = -1$

2. $y_1(0) = 0, \qquad y_1'(0) = 1, \qquad y_2(0) = -1, \qquad y_2'(0) = 0$

3.12 Higher Order Homogeneous Constant Coefficient Differential Equations

Consider: $(-1)^{1/3} = -1$

What about the solutions to $x^3 + 1 = 0$? There are three, so where are the other two?

Try $x = \dfrac{1 + i\sqrt{3}}{2}$: $\left(\dfrac{1 + i\sqrt{3}}{2}\right)^3 + 1 = 0$

So the third solution is

In order to find these solutions in the complex plane you need to start with Eulers formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For this example we can write -1 as a complex number:

$$-1 = e^{\pi i} = \cos\pi + i\sin\pi$$

But we could have also written this more generically:

$$-1 = e^{(\pi + 2n\pi)i}, \qquad n \in \mathbb{Z}$$

So if we want three solutions to $(-1)^{1/3}$ we can use the complex form:

$$(-1)^{1/3} = \left(e^{(\pi+2n\pi)}\right)^{1/3} = e^{(\pi/3 + 2n\pi/3)}, \qquad n = 0, 1, 2$$

which provides three distinct answers:

$$(-1)^{1/3} = e^{\pi/3i}, e^{\pi i}, e^{5\pi/3i}.$$

These can be graphed in the Real-Imaginary plane:

We can use this to solve differential equations of higher order.

Example 3.12.1. Solve $y^{(4)} - 8y' = 0$

The procedure is the same as for second order equations. We assume that our solution looks like $y = e^{r_i t}$ where r_i is a solution to the characteristic equation:

$$r^4 - 8r = 0$$

$$r(r^3 - 8) = 0$$

so either $r = 0$ or $r^3 - 8 = 0$

The 4 solutions are $r_1 = 0$, $r_1 = 2$, $r_1 = -1 + \sqrt{3}i$, $r_1 = -1 - \sqrt{3}i$ and the complete solution is:

Example 3.12.2. The 4th order differential equation $y^{(4)} - y''' + 8y'' - 8y' + 4y = 0$ has characteristic equation $(r - (1 + i))^2(r - (1 - i))^2 = 0$ in factored form. What is the general solution?

Example 3.12.3. Solve the IVP: $y^{(4)} - y''' = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 1$