10.1 Counting by Systematic Listing

In this chapter we will be looking at different ways of counting outcomes. We will be given a set of items and we want to know the number of ways to arrange them.

Example 1

1. List the results of tossing a coin.
2. List the results of rolling a die.

Both of these examples represent a **one-part task**.

Example 2 Consider the club with 6 members:

$$\{\text{Vicki, Luis, Maddy, Todd, Ron, Ben}\}$$

How many ways are there of selecting a president?

Two-part tasks are more difficult.

Example 3 How many ways are there for the club in Example 2 to select both a president and a secretary? We will assume that no one can hold more than one office at a time.
To solve this problem we will construct a tree diagram listing all the possible outcomes. We will abbreviate the club members as {V, L, M, T, R, B}. Since there are two things we are trying to count, president and secretary, we can construct a table:

Example 4 Determine the number of different possible results when two ordinary dice are rolled. We are assuming here that the dice are distinguishable in some way. Suppose one is red and one is green.

Example 5 Determine the number of different possible results when two ordinary dice are rolled are exactly 9. Are less than 5. Are exactly 7.

For **tasks with three or more parts** we will use tree diagrams.

Example 6 Determine the number of ways that the club from Example 2 can choose a president, vice president and secretary.

Example 7 Determine the number of ways that the club from Example 2 can choose a president, vice president and secretary but now the secretary must be a man.

Example 8 Determine the number of ways that the club from Example 2 can choose a president, vice president and secretary but now the president must be woman and the other two must be men.

Example 9 Construct a tree diagram showing all possible results when three fair coins are tossed. Then list the ways of getting the following results.

1. at least two heads
2. more than two heads
3. no more than two heads
4. fewer than two heads

Example 10 How many integers between 400 and 899 contain the digit 8?

Example 11 A group of 24 strangers sat in a circle, and each got acquainted only with the person to the left and the person to the right. Then all 24 people stood up and each one shook hands (once) with each of the others who was still a stranger. How many handshakes occurred?

10.2 Using the Fundamental Counting Principle

A multi-part task is said to satisfy the **uniformity criterion** if the number of choices for any particular part is the same *no matter which choices were selected for previous parts.*

Fundamental Counting Principle

When a task consists of k separate parts and satisfies the uniformity criterion, if the first part can be none in n_1 ways, the second part can then be done in n_2 ways, and so on through the k_{th} part, which can be done in n_k ways, then the total number of ways to complete the task is given by the product

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k.$$

Factorial Formula

For any counting number n, the quantity n factorial is given by

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1.$$

Definition of Zero Factorial

$$0! = 1$$

(because it makes our other calculations easier.)

Arrangement of n Objects

The total number of different ways to arrange n distinct objects is $n!$.
Examples:

1. How many nonrepeating odd three digit numbers are there?

2. For the following problems counting numbers are to be formed using only digits from the set 3, 4, 5. Determine the number of different possibilities for each type of number described.

   (a) Two digit numbers

   (b) odd three digit numbers

   (c) four-digit numbers with one pair of adjacent 4s and no other repeated digits.

   (d) five-digit numbers beginning and ending with 3 and with unlimited repetitions allowed.

3. The Casa Loma Restaurant offers four choices in the soup and salad category (two soups and two salads), two choices in the bread category, and three choices in the entree category. Find the number of dinners available in each of the following cases.

   (a) One item is to be included from each of the three categories.

   (b) only soup and entree are to be included.

4. Determine the number of possible ways to mark your answer sheet (with an answer for each question) for each of the following tests.

   (a) a six-question true-or-false test

   (b) a twenty-question multiple-choice test with five answer choices for each question.

Arrangement of n Objects Containing Look-Alikes

The number of **distinguishable arrangements** of n objects, where one or more subsets consists of look-alikes (say n_1 are of one kind and n_2 are of another kind, ..., and n_k are of yet another kind), is given by

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Examples: Distinguishable arrangements of the letters in:

SYNDICATE

GOOGOL

HEEBIE-JEEBIES

10.3 Using Permutations and Combinations

Permutation Formula

The number of **permutations,** or *arrangements*, of n distinct things taken r at a time, where $r \leq n$, is given by

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

This is a difficult way to remember the formula so we usually think of it this way

$${}_nP_r = \frac{n!}{(n-r)!}.$$

When do we use the permutation formula?

When repetitions are not allowed AND order matters.

Combination Formula

The number of **combinations,** or *subsets*, of n distinct things taken r at a time, where $r \leq n$, is given by

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 2\cdot 1}.$$

This is a difficult way to remember the formula so we usually think of it this way

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

When do we use the combination formula?

When repetitions are not allowed AND order is **not** important.

Guidelines for Choosing a Counting Method

1. If selected items can be repeated, use fundamental counting principle.
   Example: How many 4 digit numbers are there?

   $$9\cdot 10\cdot 10\cdot 10 = 9000$$

2. if selected items cannot be repeated, and order is important, use permutations.
   Example: How many ways can three of eight people line up at a ticket counter?

   $${}_8P_3 = 8\cdot 7\cdot 6 = 336$$

3. If selected items cannot be repeated, and order is not important, use combinations.
   Example: How many ways can a committee of three people be selected from a group of 12 people?

   $${}_{12}C_3 = \frac{12!}{3!9!} = 220$$

The following table summarizes the differences:

Permutations	**Combinations**
Number of ways of selecting r items out of n items	
Repetitions are not allowed	
Order is important	Order is not important
Arrangements on n items taken r at a time	Subsets of n items taken r at a time
${}_nP_r = \frac{n!}{(n-r)!}$	${}_nC_r = \frac{n!}{r!(n-r)!}$
Clue words: Arrangement, schedule, order	Clue words: set, group, sample, selection

Examples

1. Decide whether each object is a permutation or a combination.
   (a) a telephone number
   (b) a social security number
   (c) a hand of cards in poker
   (d) a committee of politicians
   (e) the "‘combination”’ on a combination lock
   (f) a lottery choice of six numbers where the order does not matter
   (g) an automobile license plate number

2. Jeff Hubbard, a contractor, builds homes of eight different models and presently has five lots to build on. In how many different ways can he arrange homes on these lots? Assume five different models will be built.

3. How many ways can a teacher give five different prizes to five of her 25 students?

4. How many 5 member committees could be formed from the 100 U.S. senators?

5. A standard 52 card deck contains 4 aces, twelve face cards, thirteen hearts (all red), thirteen diamonds (all red), thirteen spades (all black), and thirteen clubs (all black). Of the 2,598,960 different five-card hands possible, decide how many would consist of the following cards

   (a) only cards of a single suit. (ie. a flush)

   (b) all red cards

   (c) 4 aces and any other card

6. The Coyotes, a youth league baseball team, have seven pitchers, who only pitch, and twelve other players, all of whom can play any position other than pitcher. For Saturday's game, the coach has not yet determined which nine players to use nor what the batting order will be, except that the pitcher will bat last. How many different batting orders may occur?

7. Nine people are to be distributed among three committees of two, three, and four members and a chairperson is to be selected for each committee. Use the following steps to determine how many ways this be done.

   (a) Select the member of the two-person committee

   (b) Select the members of the three-person committee

   (c) Select the chair for the two-person committee

(d) Select the chair for the three-person committee

(e) Select the chair for the four-person committee

8. Because of his good work Jeff Hubbard gets a contract to build homes on three additional blocks in the subdivision, with six homes on each block. He decides to build nine deluxe homes on these three blocks: two on the first block, three on the second, and four on the third. The remaining 9 homes will be standard.

   (a) Altogether on the three-block stretch, how many different choices does Jeff have for positioning the 18 homes?

   (b) How many choices would he have if he built 2, 3 and 4 deluxe models on the three different blocks as before, but not necessarily on the first second and third blocks in that order?

10.4 Using Pascal's Triangle and the Binomial Theorem

Very often the notation ${}_nC_r$ is written as $\binom{n}{r}$ and we read it as "n choose r ".

Binomial Theorem

For any positive integer n,

$$\begin{aligned}(x+y)^n &= {}_nC_0 \cdot x^n +_n C_1 \cdot x^{n-1}y +_n C_2 \cdot x^{n-2}y^2 +_n C_3 \cdot x^{n-3}y^3 + \\ &\quad \ldots +_n C_{n-1} \cdot xy^{n-1} +_n C_n \cdot y^n\end{aligned}$$

This is a difficult way to remember the formula so we usually think of it this way

$$\begin{aligned}(x+y)^n &= x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \binom{n}{3} \cdot x^{n-3}y^3 + \\ &\quad \ldots + \binom{n}{n-1} \cdot xy^{n-1} + \cdot y^n\end{aligned}$$

Pascal's Triangle

See handout

Examples

1. Read the following combination values directly from Pascal's triangle.
   (a) ${}_4C_3$
   (b) ${}_6C_4$
   (c) ${}_9C_7$
2. A committee of four Congressmen will be selected from a group of seven Democrats and three Republicans. Find the number of ways of obtainin each of the following.
   (a) Exactly one Democrat
   (b) Exactly two Democrats
   (c) Exactly three Democrats
   (d) Exactly four Democrats

3. Suppose eight fair coins are tossed. Find the number of ways of obtaining exactly 3 heads. Exactly 5 heads.

4. Expand the following

   (a) $(p+q)^4$

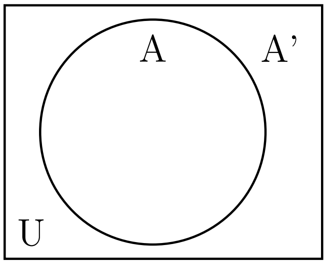
   (b) $(p-q)^4$

5. Find the 6th term of $(x+y)^9$

10.5 Counting Problems involving "Not" and "Or"

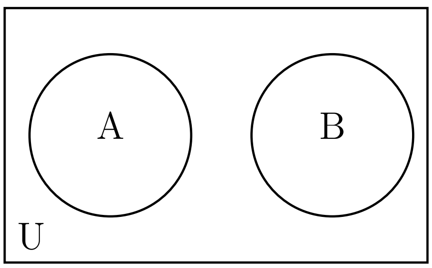
Complement Principle of Counting: If A is any set within the universal set U, then

$$n(A) = n(U) - n(A')$$



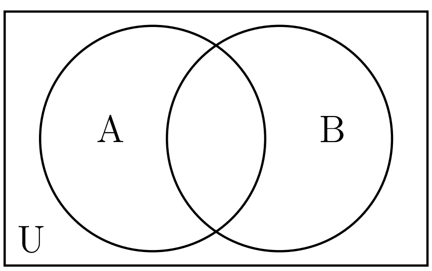
The Special Additive Principle: For any two disjoint sets A and B

$$n(A \cup B) = n(A) + n(B)$$



General Additive Counting Principle: If A and B are any two sets, disjoint or not, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



1. For the set {A, B, C, D, E} how many proper subsets are there?

2. If you toss 7 fair coins, in how many ways can you obtain at least one head.

3. If you roll two fair dice (say red and green), in how many ways can you obtain a 4 on at least one of the dice.

4. How many two digit counting numbers are not a multiple of 10.

5. The ten 10 longest Broadway runs include *Cats* and *Oh, Calcultta!*. Four of the ten are chosen randomly. (Assume order is not important)

   (a) How many ways can the four be chosen?

   (b) How many of those groups would include neither of the productions mentioned?

   (c) How many of them would include at least one of the two productions mentioned?

6. A Civil Air Patrol unit of twelve members includes four officers. In how many ways can four members be selected for a search and rescue mission such that at least one officer is included?

7. Among the 2,598,960 possible 5-card poker hands from a standard 52-card deck, how many contain at least one king.